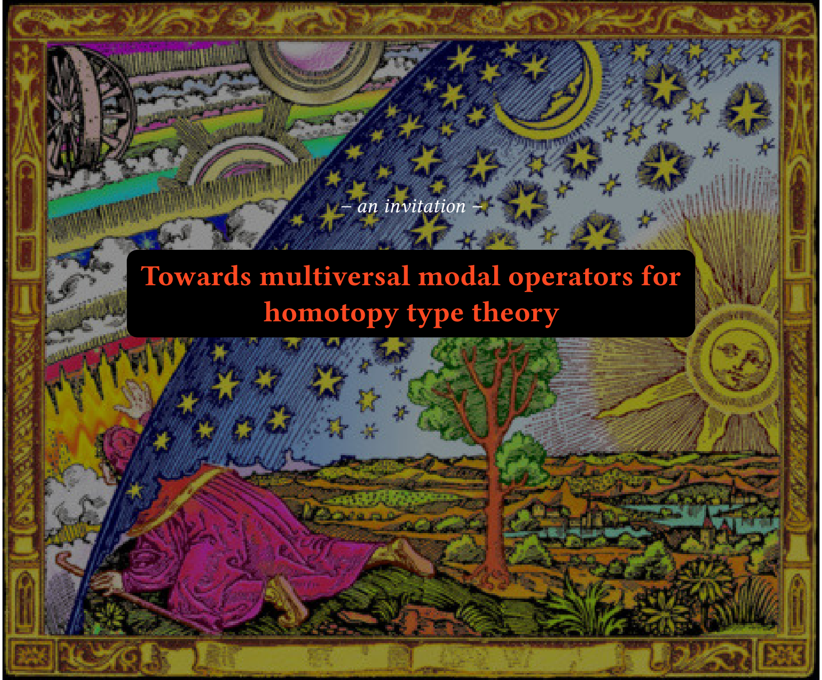


– an invitation –

Towards multiversal modal operators for homotopy type theory

How should this notion be formalized in HoTT?

$$7, \quad 4, \quad 3, \quad \ldots$$

How should this notion be formalized in HoTT?

$$7, \quad 4, \quad 3, \quad 1, \quad \ldots$$

How should this notion be formalized in HoTT?

$$7, \quad 4, \quad 3, \quad 1, \quad 8, \quad \ldots$$

$\leq$

How should this notion be formalized in HoTT?

$$7, \quad 4, \quad 3, \quad 1, \quad 8, \quad 2, \quad \ldots$$

$$3 \leq 8$$

Thm. Every sequence $\alpha : \mathbb{N} \to \mathbb{N}$ is **good** in that there exist $i < j$ with $\alpha\, i \leq \alpha\, j$.

How should this notion be formalized in HoTT?

$$7, \quad 4, \quad 3, \quad 1, \quad 8, \quad 2, \quad \dots$$

$$3 \leq 8$$

Thm. Every sequence $\alpha : \mathbb{N} \to \mathbb{N}$ is **good** in that there exist $i < j$ with $\alpha\, i \leq \alpha\, j$.

Proof. (offensive?) By LEM, there is a minimum $\alpha\, i$. Set $j := i + 1$. □

How should this notion be formalized in HoTT?

$$7, \quad 4, \quad 3, \quad 1, \quad 8, \quad 2, \quad \ldots$$

$\leq$

Thm. Every sequence $\alpha : \mathbb{N} \to \mathbb{N}$ is **good** in that there exist $i < j$ with $\alpha\, i \leq \alpha\, j$.

Proof. (offensive?) By LEM, there is a minimum $\alpha\, i$. Set $j := i + 1$. □

Def. (classically) A quasiorder X is **well** iff every sequence $\mathbb{N} \to X$ is good.

Examples. (classically) $(\mathbb{N}, \leq)$, $X \times Y$, X^*, $\mathrm{Tree}(X)$.

How should this notion be formalized in HoTT?

$$7, \quad 4, \quad 3, \quad 1, \quad 8, \quad 2, \quad \ldots$$

$$3 \leq 8$$

Thm. Every sequence $\alpha : \mathbb{N} \to \mathbb{N}$ is **good** in that there exist $i < j$ with $\alpha\, i \leq \alpha\, j$.

Proof. (offensive?) By LEM, there is a minimum $\alpha\, i$. Set $j := i + 1$. □

Def. (classically) A quasiorder X is **well** iff every sequence $\mathbb{N} \to X$ is good.

Examples. (classically) $(\mathbb{N}, \leq)$, $X \times Y$, X^*, $\mathrm{Tree}(X)$.

A naive formalization attempt:

Def. $\mathrm{Well}_\infty(X, \leq) := \prod_{\alpha : \mathbb{N} \to X} \left\| \sum_{i : \mathbb{N}} \sum_{j : \mathbb{N}} i < j \times \alpha\, i \leq \alpha\, j \right\|_{-1}$

How should this notion be formalized in HoTT?

$7, \quad 4, \quad 3, \quad 1, \quad 8, \quad 2, \quad \ldots$

$\leq$

Thm. Every sequence $\alpha : \mathbb{N} \to \mathbb{N}$ is **good** in that there exist $i < j$ with $\alpha\, i \leq \alpha\, j$.

Proof. (offensive?) By LEM, there is a minimum $\alpha\, i$. Set $j := i + 1$. □

Def. (classically) A quasiorder X is **well** iff every sequence $\mathbb{N} \to X$ is good.

Examples. (classically) $(\mathbb{N}, \leq)$, $X \times Y$, X^*, $\mathrm{Tree}(X)$.

A naive formalization attempt:

Def. $\mathrm{Well}_\infty(X, \leq) :\equiv \prod_{\alpha : \mathbb{N} \to X} \left\| \sum_{i : \mathbb{N}} \sum_{j : \mathbb{N}} i < j \times \alpha\, i \leq \alpha\, j \right\|_{-1}$

- ✗ philosophically strenuous
- ✗ not practical
- ✗ not faithful(?)

Sequences provided by LEM

Lemma. Let X be a well quasiorder. Let $\alpha : \mathbb{N} \to X$ be a sequence. Assuming LEM, there merely is a monotonic subsequence $\alpha\, i_0 \leq \alpha\, i_1 \leq \cdots$.

Proof. The type

$$I := \sum_{i:\mathbb{N}} \neg \sum_{j:\mathbb{N}} i < j \times \alpha\, i \leq \alpha\, j$$

cannot be in bijection with $\mathbb{N}$, as else the I-extracted subsequence of α would not be good. By LEM, the type I is finite. Any index i_0 larger than all the indices in I is a suitable starting point for a monotonic subsequence. □

Sequences provided by LEM

Lemma. Let X be a well quasiorder. Let $\alpha : \mathbb{N} \to X$ be a sequence. Assuming LEM, there merely is a monotonic subsequence $\alpha\ i_0 \leq \alpha\ i_1 \leq \cdots$.

Proof. The type

$$I := \sum_{i:\mathbb{N}} \neg \sum_{j:\mathbb{N}} i < j \times \alpha\ i \leq \alpha\ j$$

cannot be in bijection with $\mathbb{N}$, as else the I-extracted subsequence of α would not be good. By LEM, the type I is finite. Any index i_0 larger than all the indices in I is a suitable starting point for a monotonic subsequence. □

Prop. Let X and Y be well quasiorders. Assuming LEM, $X \times Y$ is well.

Proof. Let an infinite sequence $\gamma : \mathbb{N} \to X \times Y$ be given. Write $\gamma\ k = (\alpha\ k, \beta\ k)$. By the lemma, there is a monotonic subsequence $\alpha\ i_0 \leq \alpha\ i_1 \leq \cdots$. Because Y is well, there are indices $n < m$ such that $\beta\ i_n \leq \beta\ i_m$. As also $\alpha\ i_n \leq \alpha\ i_m$, the sequence γ is good. □

Sequences provided by LEM

Lemma. Let X be a well quasiorder. Let $\alpha : \mathbb{N} \to X$ be a sequence. Assuming LEM, there merely is a monotonic subsequence $\alpha\, i_0 \leq \alpha\, i_1 \leq \cdots$.

Proof. The type

$$I := \sum_{i:\mathbb{N}} \neg \sum_{j:\mathbb{N}} i < j \times \alpha\, i \leq \alpha\, j$$

cannot be in bijection with $\mathbb{N}$, as else the I-extracted subsequence of α would not be good. By LEM, the type I is finite. Any index i_0 larger than all the indices in I is a suitable starting point for a monotonic subsequence. □

Prop. Let X and Y be well quasiorders. Assuming LEM, $X \times Y$ is well.

Proof. Let an infinite sequence $\gamma : \mathbb{N} \to X \times Y$ be given. Write $\gamma\, k = (\alpha\, k, \beta\, k)$. By the lemma, there is a monotonic subsequence $\alpha\, i_0 \leq \alpha\, i_1 \leq \cdots$. Because Y is well, there are indices $n < m$ such that $\beta\, i_n \leq \beta\, i_m$. As also $\alpha\, i_n \leq \alpha\, i_m$, the sequence γ is good. □

We cannot trust LEM-provided sequences to be available in the type $\mathbb{N} \to X$. Similarly with DC.

Sequences depending on an environment

Let X be an hset such that there is no surjection $\mathbb{N} \twoheadrightarrow X$. Then the type $\mathbb{N} \to X$ misses the **generic enumeration** α of X.

Sequences depending on an environment

Let X be an hset such that there is no surjection $\mathbb{N} \twoheadrightarrow X$. Then the type $\mathbb{N} \to X$ misses the **generic enumeration** α of X.

Idea: Approximate (fictitious) surjections $\mathbb{N} \twoheadrightarrow X$ by finite sequences $\sigma = x_0 \ldots x_{n-1} : X^*$. Starting with the empty sequence ε, over time, such an approximation can grow to

1. one of σy, where $y : X$, so it becomes *more defined*; or
2. one of $\sigma\tau$, where $\tau : X^*$ such that $a \in \sigma\tau$, so it becomes *more surjective*.

Sequences depending on an environment

Let X be an hset such that there is no surjection $\mathbb{N} \twoheadrightarrow X$. Then the type $\mathbb{N} \to X$ misses the **generic enumeration** α of X.

Idea: Approximate (fictitious) surjections $\mathbb{N} \twoheadrightarrow X$ by finite sequences $\sigma = x_0 \ldots x_{n-1} : X^*$. Starting with the empty sequence ε, over time, such an approximation can grow to

1. one of σy, where $y : X$, so it becomes *more defined*; or
2. one of $\sigma\tau$, where $\tau : X^*$ such that $a \in \sigma\tau$, so it becomes *more surjective.*

For $k : \mathbb{N}$ and $a : X$, the expression "$\alpha\, k = a$" does not denote a proposition but rather a **stage-dependent proposition** $X^* \to \mathrm{Prop}$, namely $\lambda x_0 \ldots x_{n-1}.\ k < n \times x_k = a$.

Sequences depending on an environment

Let X be an hset such that there is no surjection $\mathbb{N} \twoheadrightarrow X$. Then the type $\mathbb{N} \to X$ misses the **generic enumeration** α of X.

Idea: Approximate (fictitious) surjections $\mathbb{N} \twoheadrightarrow X$ by finite sequences $\sigma = x_0 \ldots x_{n-1} : X^*$. Starting with the empty sequence ε, over time, such an approximation can grow to

1. one of σy, where $y : X$, so it becomes *more defined*; or
2. one of $\sigma\tau$, where $\tau : X^*$ such that $a \in \sigma\tau$, so it becomes *more surjective*.

For $k : \mathbb{N}$ and $a : X$, the expression "$\alpha\, k = a$" does not denote a proposition but rather a **stage-dependent proposition** $X^* \to \mathrm{Prop}$, namely $\lambda x_0 \ldots x_{n-1}.\, k < n \times x_k = a$.

Given a stage-dependent proposition P, $\nabla P\, \sigma$ expresses that no matter how σ evolves to a better approximation τ, eventually $P\,\tau$ will hold:

$$
\begin{aligned}
&\mathsf{data}\ \nabla\ (P : \mathsf{X}^* \to \mathsf{Prop})\ (\sigma : \mathsf{X}^*) : \mathsf{Prop}\ \mathsf{where} \\
&\quad \mathsf{now} : P\,\sigma \to \nabla\ P\,\sigma \\
&\quad \mathsf{later}_1 : ((y : \mathsf{X}) \to \nabla\ P\,(\sigma y)) \to \nabla\ P\,\sigma \\
&\quad \mathsf{later}_2 : (a : \mathsf{X}) \to ((\tau : \mathsf{X}^*) \to a \in \sigma\tau \to \nabla\ P\,(\sigma\tau)) \to \nabla\ P\,\sigma
\end{aligned}
$$

For instance, for $P_a\,\sigma :\equiv (a \in \sigma)$, we have $\nabla P_a\,\varepsilon$. "α is surjective": $(a : \mathsf{X}) \to \nabla\ P_a\,\varepsilon$

A naive formalization attempt:

Def. $\mathsf{Well}_\infty(X, \leq) :\equiv \prod_{\alpha\,:\,\mathbb{N}\to X} \left\| \sum_{i\,:\,\mathbb{N}} \sum_{j\,:\,\mathbb{N}} i < j \times \alpha\, i \leq \alpha\, j \right\|_{-1}$

Well quasiorders revisited

A naive formalization attempt:

Def. $\mathrm{Well}_\infty(X, \leq) :\equiv \prod_{\alpha : \mathbb{N} \to X} \left\| \sum_{i : \mathbb{N}} \sum_{j : \mathbb{N}} i < j \times \alpha\, i \leq \alpha\, j \right\|_{-1}$

An inductive rephrasing:

Def. $\mathrm{Well}(X, \leq) :\equiv \nabla \mathrm{Good}\, \varepsilon$, where $\mathrm{Good}\, x_0 \ldots x_{n-1} :\equiv \exists_{i:\mathbb{N}} \exists_{j:\mathbb{N}} (i < j \times x_i \leq x_j)$ and

```
data ∇ (P : X* → Prop) (σ : X*) : Prop where
  now : P σ → ∇ P σ
  later : ((y : X) → ∇ P (σ y)) → ∇ P σ
```

In other words: *A quasiorder X is well iff the generic sequence $\mathbb{N} \to X$ is good.*

Prop. Let X and Y be well quasiorders. Without assuming LEM, $X \times Y$ is well.

Well quasiorders revisited

A naive formalization attempt:

Def. $\mathrm{Well}_\infty(X, \leq) :\equiv \prod_{\alpha : \mathbb{N} \to X} \left\| \sum_{i : \mathbb{N}} \sum_{j : \mathbb{N}} i < j \times \alpha\, i \leq \alpha\, j \right\|_{-1}$

An inductive rephrasing:

Def. $\mathrm{Well}(X, \leq) :\equiv \nabla \mathrm{Good}\, \varepsilon$, where $\mathrm{Good}\, x_0 \ldots x_{n-1} :\equiv \exists_{i:\mathbb{N}} \exists_{j:\mathbb{N}} (i < j \times x_i \leq x_j)$ and

```
data ∇ (P : X* → Prop) (σ : X*) : Prop where
  now : P σ → ∇ P σ
  later : ((y : X) → ∇ P (σ y)) → ∇ P σ
```

In other words: *A quasiorder X is well iff the generic sequence $\mathbb{N} \to X$ is good.*

Prop. Let X and Y be well quasiorders. Without assuming LEM, $X \times Y$ is well.

We have $\mathrm{Well}(X, \leq) \to \mathrm{Well}_\infty(X, \leq)$, but the converse only holds in presence of **bar induction**. *How much stronger exactly is the inductive rephrasing?*

Topos for reifying generic models

Grothendieck toposes (= categories of sheaves over sites) are mathematical universes:

- The **generic sequence** is an honest function $\mathbb{N} \to X$ **in** a certain topos $\mathcal{E}$.
- The **generic surjection** is an honest surjection **in** an appropriate subtopos of $\mathcal{E}$.
- In the *double-negation subtopos* of the base, a predicate $Q : \mathbb{N} \to \mathrm{Prop}$ such that

  $$Q\,x \times Q\,y \longrightarrow x = y \quad \text{and} \quad \neg\neg \sum_{n : \mathbb{N}} Q\,n$$

  looks like an ordinary natural number.

A statement holds in a topos iff a certain topos-directed translation holds in the base. For instance, for the double-negation subtopos the translation substitutes

$$\begin{array}{llllll} \exists & \rightsquigarrow & \exists^{\mathrm{cl}}, & \text{where} & \exists^{\mathrm{cl}} & :\equiv \neg\neg\exists, \\ \vee & \rightsquigarrow & \vee^{\mathrm{cl}}, & \text{where} & \alpha \vee^{\mathrm{cl}} \beta & :\equiv \neg\neg(\alpha \vee \beta), \\ = & \rightsquigarrow & =^{\mathrm{cl}}, & \text{where} & s =^{\mathrm{cl}} t & :\equiv \neg\neg(s = t). \end{array}$$

In general, stage-dependent ∇ instead of $\neg\neg$.

Maximal ideals as convenient fictions

Thm. Let M be a surjective matrix with more rows than columns over a commutative ring A. Then $1 = 0$ in A.

Maximal ideals as convenient fictions

Thm. Let M be a surjective matrix with more rows than columns over a commutative ring A. Then $1 = 0$ in A.

Proof. (classical) Assume not.

Maximal ideals as convenient fictions

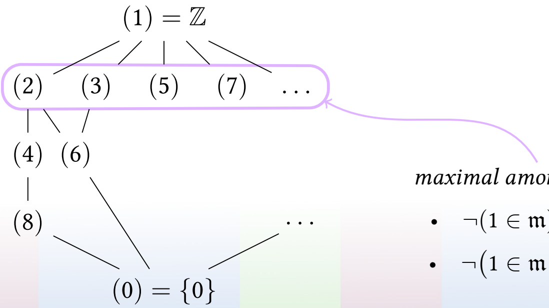
Thm. Let M be a surjective matrix with more rows than columns over a commutative ring A. Then $1 = 0$ in A.

Proof. (classical) Assume not. Then there is a maximal ideal $\mathfrak{m}$.

Maximal ideals as convenient fictions

Thm. Let M be a surjective matrix with more rows than columns over a commutative ring A. Then $1 = 0$ in A.

Proof. (classical) Assume not. Then there is a maximal ideal $\mathfrak{m}$.



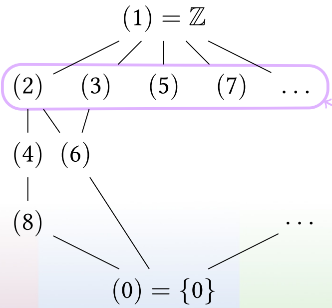
maximal among the proper ideals

- $\neg(1 \in \mathfrak{m})$
- $\neg\big(1 \in \mathfrak{m} + (x)\big) \Rightarrow x \in \mathfrak{m}$

Maximal ideals as convenient fictions

Thm. Let M be a surjective matrix with more rows than columns over a commutative ring A. Then $1 = 0$ in A.

Proof. (classical) Assume not. Then there is a maximal ideal $\mathfrak{m}$. The matrix is surjective over $A/\mathfrak{m}$.



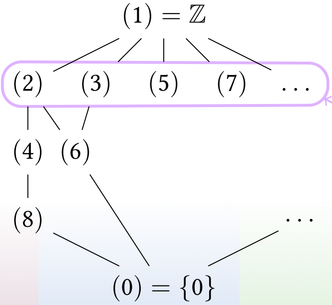
maximal among the proper ideals

- $\neg(1 \in \mathfrak{m})$
- $\neg\big(1 \in \mathfrak{m} + (x)\big) \Rightarrow x \in \mathfrak{m}$

Maximal ideals as convenient fictions

Thm. Let M be a surjective matrix with more rows than columns over a commutative ring A. Then $1 = 0$ in A.

Proof. (classical) Assume not. Then there is a maximal ideal $\mathfrak{m}$. The matrix is surjective over $A/\mathfrak{m}$. Since $A/\mathfrak{m}$ is a field, this is a contradiction to basic linear algebra. □



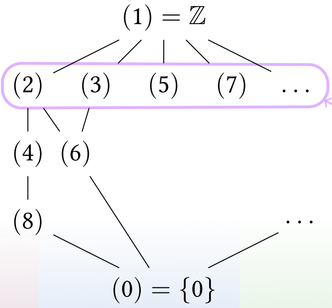
maximal among the proper ideals

- $\neg(1 \in \mathfrak{m})$
- $\neg\big(1 \in \mathfrak{m} + (x)\big) \Rightarrow x \in \mathfrak{m}$

Maximal ideals as convenient fictions

Thm. Let M be a surjective matrix with more rows than columns over a commutative ring A. Then $1 = 0$ in A.

Proof. (classical) Assume not. Then there is a maximal ideal $\mathfrak{m}$. The matrix is surjective over $A/\mathfrak{m}$. Since $A/\mathfrak{m}$ is a field, this is a contradiction to basic linear algebra. □



maximal among the proper ideals

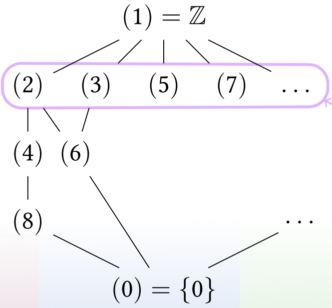
- $\neg(1 \in \mathfrak{m})$
- $\neg(1 \in \mathfrak{m} + (x)) \Rightarrow x \in \mathfrak{m}$

Does there exist a maximal ideal?

Maximal ideals as convenient fictions

Thm. Let M be a surjective matrix with more rows than columns over a commutative ring A. Then $1 = 0$ in A.

Proof. (classical) Assume not. Then there is a maximal ideal $\mathfrak{m}$. The matrix is surjective over $A/\mathfrak{m}$. Since $A/\mathfrak{m}$ is a field, this is a contradiction to basic linear algebra. □



maximal among the proper ideals

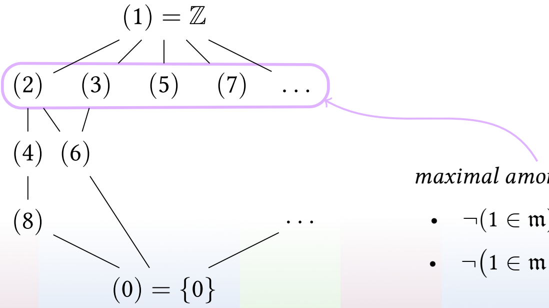
- $\neg(1 \in \mathfrak{m})$
- $\neg\big(1 \in \mathfrak{m} + (x)\big) \Rightarrow x \in \mathfrak{m}$

Does there exist a maximal ideal? Yes, if A is countable. In the general case: No

Maximal ideals as convenient fictions

Thm. Let M be a surjective matrix with more rows than columns over a commutative ring A. Then $1 = 0$ in A.

Proof. (classical) Assume not. Then there is a maximal ideal $\mathfrak{m}$. The matrix is surjective over $A/\mathfrak{m}$. Since $A/\mathfrak{m}$ is a field, this is a contradiction to basic linear algebra. □



maximal among the proper ideals

- $\neg(1 \in \mathfrak{m})$
- $\neg(1 \in \mathfrak{m} + (x)) \Rightarrow x \in \mathfrak{m}$

Does there exist a maximal ideal? Yes, if A is countable. In the general case: No, but yes in a *suitable topos*, and *bounded first-order consequences* of its existence there pass down to the base.

Maximal ideals as convenient fictions

Thm. Let M be a surjective matrix with more rows than columns over a commutative ring A. Then $1 = 0$ in A.

Proof. (constructive, special case) Write $M = \binom{x}{y}$. By surjectivity, we have $u, v : A$ with

$$u \binom{x}{y} = \binom{1}{0} \quad \text{and} \quad v \binom{x}{y} = \binom{0}{1}.$$

Hence $1 = (vy)(ux) = (uy)(vx) = 0$. □

A brief timeline

1878 · · · · · Cantor advances the **continuum hypothesis**, the claim that $2^{\aleph_0} = \aleph_1$.

A brief timeline

1878 · · · · · Cantor advances the **continuum hypothesis**, the claim that $2^{\aleph_0} = \aleph_1$.

1910s · · · · · Zermelo–Fraenkel set theory emerges.

A brief timeline

1878 · · · · · · Cantor advances the **continuum hypothesis**, the claim that $2^{\aleph_0} = \aleph_1$.

1910s · · · · · Zermelo–Fraenkel set theory emerges.

1920s · · · · · Set theorists pursue additional axioms to settle CH (one way or another).

A brief timeline

- 1878 — Cantor advances the **continuum hypothesis**, the claim that $2^{\aleph_0} = \aleph_1$.
- 1910s — Zermelo–Fraenkel set theory emerges.
- 1920s — Set theorists pursue additional axioms to settle CH (one way or another).
- 1938 — Gödel proves: If ZFC is consistent, so is ZFC+CH.

A brief timeline

1878 · · · · · Cantor advances the **continuum hypothesis**, the claim that $2^{\aleph_0} = \aleph_1$.

1910s · · · · · Zermelo–Fraenkel set theory emerges.

1920s · · · · · Set theorists pursue additional axioms to settle CH (one way or another).

1938 · · · · · Gödel proves: If ZFC is consistent, so is ZFC+CH.

1963 · · · · · Cohen proves: If ZFC is consistent, so is ZFC+¬CH.

A brief timeline

- 1878 — Cantor advances the **continuum hypothesis**, the claim that $2^{\aleph_0} = \aleph_1$.
- 1910s — Zermelo–Fraenkel set theory emerges.
- 1920s — Set theorists pursue additional axioms to settle CH (one way or another).
- 1938 — Gödel proves: If ZFC is consistent, so is ZFC+CH.
- 1963 — Cohen proves: If ZFC is consistent, so is ZFC+¬CH.
- 2011 — Hamkins offers his paper on the **multiverse position** in the philosophy of set theory.

A brief timeline

1878 · · · · · Cantor advances the **continuum hypothesis**, the claim that $2^{\aleph_0} = \aleph_1$.

1910s · · · · · Zermelo–Fraenkel set theory emerges.

1920s · · · · · Set theorists pursue additional axioms to settle CH (one way or another).

1938 · · · · · Gödel proves: If ZFC is consistent, so is ZFC+CH.

1963 · · · · · Cohen proves: If ZFC is consistent, so is ZFC+¬CH.

2011 · · · · · Hamkins offers his paper on the **multiverse position** in the philosophy of set theory.

2016 · · · · · Oldenziel proposes to study the modal multiverse of toposes.

A brief timeline

1878 · · · · · Cantor advances the **continuum hypothesis**, the claim that $2^{\aleph_0} = \aleph_1$.

1910s · · · · · Zermelo–Fraenkel set theory emerges.

1920s · · · · · Set theorists pursue additional axioms to settle CH (one way or another).

1938 · · · · · Gödel proves: If ZFC is consistent, so is ZFC+CH.

1963 · · · · · Cohen proves: If ZFC is consistent, so is ZFC+¬CH.

2011 · · · · · Hamkins offers his paper on the **multiverse position** in the philosophy of set theory.

2016 · · · · · Oldenziel proposes to study the modal multiverse of toposes.

Judith Roitman

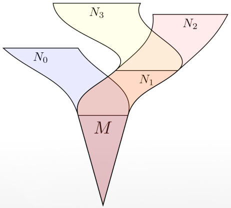
Mainstream mathematics is beginning to see results using modern set theoretic techniques.

The modal set-theoretic multiverse

Def. A **model of set theory** is a (perhaps class-sized) structure $(M, \in)$ satisfying axioms such as those of ZFC.

Examples.

- V, the class of all sets
- L, Gödel's constructible universe
- $V[G]$, a forcing extension containing a generic filter G of some poset of forcing conditions
- Henkin/term models from consistency of (extensions of) ZFC

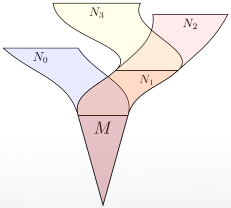


The modal set-theoretic multiverse

Def. A **model of set theory** is a (perhaps class-sized) structure $(M, \in)$ satisfying axioms such as those of ZFC.

Examples.

- V, the class of all sets
- L, Gödel's constructible universe
- $V[G]$, a forcing extension containing a generic filter G of some poset of forcing conditions
- Henkin/term models from consistency of (extensions of) ZFC



We embrace all models of set theory:

Def. $\Diamond\varphi$ iff φ holds in **some extension** of the current universe.
$\Box\varphi$ iff φ holds in **all extensions** of the current universe.

- $\Box(\Diamond \mathrm{CH} \wedge \Diamond \neg\mathrm{CH})$, the continuum hypothesis is a **switch**.
- $\Box\Diamond\Box(X \text{ is countable})$, existence of an enumeration is a **button**.

The modal topos-theoretic multiverse

Def. A statement φ holds …

1. **everywhere** ($\Box\,\varphi$) iff it holds in every topos (over the current base).
2. **somewhere** ($\Diamond\,\varphi$) iff it holds in some positive topos.
3. **proximally** ($\diamondsuit\!\!\!\!\!\diamond\,\varphi$) iff it holds in some positive overt topos.

The modal topos-theoretic multiverse

Def. A statement φ holds …

1. **everywhere** ($\Box\varphi$) iff it holds in every topos (over the current base).
2. **somewhere** ($\Diamond\varphi$) iff it holds in some positive topos.
3. **proximally** ($\diamondsuit\varphi$) iff it holds in some positive overt topos.

Traveling the multiverse:

- For every inhabited set X, *proximally* there is an enumeration $\mathbb{N} \twoheadrightarrow X$.
- A quasiorder is well iff *everywhere*, every sequence is good.
- A ring element is nilpotent iff all prime ideals *everywhere* contain it.
- For every ring, *proximally* there is a maximal ideal.
- A relation is well-founded iff *everywhere*, there is no descending chain.
- *Somewhere*, the law of excluded middle holds.

The modal topos-theoretic multiverse

Def. A statement φ holds …

1. **everywhere** ($\Box\varphi$) iff it holds in every topos (over the current base).
2. **somewhere** ($\Diamond\varphi$) iff it holds in some positive topos.
3. **proximally** ($\diamondsuit\varphi$) iff it holds in some positive overt topos.

Traveling the multiverse:

- For every inhabited set X, *proximally* there is an enumeration $\mathbb{N} \twoheadrightarrow X$.
- A quasiorder is well iff *everywhere*, every sequence is good.
- A ring element is nilpotent iff all prime ideals *everywhere* contain it.
- For every ring, *proximally* there is a maximal ideal.
- A relation is well-founded iff *everywhere*, there is no descending chain.
- *Somewhere*, the law of excluded middle holds.

Prop. Let $(X, \leq)$ be a well quasiorder. Then $(<)$, where $x < y \equiv (x \leq y \wedge \neg(y \leq x))$, is well-founded.

Proof. Everywhere, there can be no infinite descending chain, as any such would also be good. □

Unrolling this proof gives a program $\nabla \mathrm{Good}\,\varepsilon \to \prod_{x:X} \mathrm{Acc}\,x$.

The modal topos-theoretic multiverse

Def. A statement φ holds …

1. **everywhere** ($\Box\varphi$) iff it holds in every topos (over the current base).
2. **somewhere** ($\Diamond\varphi$) iff it holds in some positive topos.
3. **proximally** ($\diamondsuit\!\!\!\Diamond\,\varphi$) iff it holds in some positive overt topos.

Traveling the multiverse:

- For every inhabited set X, *proximally* there is an enumeration $\mathbb{N} \twoheadrightarrow X$.
- A quasiorder is well iff *everywhere*, every sequence is good.
- A ring element is nilpotent iff all prime ideals *everywhere* contain it.
- For every ring, *proximally* there is a maximal ideal.
- A relation is well-founded iff *everywhere*, there is no descending chain.
- *Somewhere*, the law of excluded middle holds.

Foreshadowed by:

1984 André Joyal, Miles Tierney. "An extension of the Galois theory of Grothendieck".
1987 Andreas Blass. "Well-ordering and induction in intuitionistic logic and topoi".
2010s Milly Maietti, Steve Vickers. Ongoing work on arithmetic universes.
2011 Joel David Hamkins. "The set-theoretic multiverse".
2013 Shawn Henry. "Classifying topoi and preservation of higher order logic by geometric morphisms".

Towards the modal type-theoretic multiverse

✓ *There are type-theoretic multiverses,* such as

- the collection of all $\mathrm{PSh}(\mathcal{C} \times \mathcal{B})$, where $\mathcal{B}$ ranges over cube categories and $\mathcal{C}$ over arbitrary small categories, and their corresponding sheaf models

  Coquand. "A survey of constructive presheaf models of univalence". *ACM SIGLOG News*, 5.3 (2018).

Towards the modal type-theoretic multiverse

✓ *There are type-theoretic multiverses,* such as

- the collection of all $\mathrm{PSh}(\mathcal{C} \times \mathcal{B})$, where $\mathcal{B}$ ranges over cube categories and $\mathcal{C}$ over arbitrary small categories, and their corresponding sheaf models

  Coquand. "A survey of constructive presheaf models of univalence". *ACM SIGLOG News*, 5.3 (2018).

✗ *Accessing the multiverse from within type theory is tricky:*

- Given a model of $\mathfrak{s}$CIC and a category $\mathcal{C}$ in it, we have a syntactic presheaf model of CIC.

  Coquand, Jaber. "A note on forcing and type theory". *Fundamenta Informaticae 100* (2010).
  Jaber, Lewertowski, Pédrot, Sozeau, Tabareau. "The definitional side of the forcing". *Proceedings of LICS '16* (2016).
  Pédrot. "Russian constructivism in a prefascist theory". *Proceedings of LICS '20* (2020).

- Given a suitable lex modality, we have a syntactic sheaf model (model of modal types).

  Coquand, Ruch, Sattler. "Constructive sheaf models of type theory." *Math. Struct. Comput. Sci.* 31.9 (2021).
  Escardó, Xu. "Sheaf models of type theory in type theory". Unpublished (2016).
  Quirin. "Lawvere–Tierney sheafification in Homotopy Type Theory". PhD thesis (2016).

- (I believe) we have syntactic sheaf models in certain special cases, when no coherence issues arise in defining the notion of presheaves.

Still, can pragmatically use ∇ and feel philosophically inspired.