Synthetic category theory in CaTT

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HoTTEST seminar talk.

Synthetic category theory: why?

- Homotopy theory "=" $(\infty, 1)$-category theory.
- Trend towards model-agnostic arguments.
- [Cisinski-C.-Nguyen-Walde]: Build basic theory on model-agnostic language.
- Can do stable homotopy theory, higher algebra, ∞-topoi, ...
- Type theory potentially closer to model-agnostic language.

Goal: A type theory for $(\infty, 1)$-categories

Synthetic category theory: how?

Two prominent approaches, both extending Martin-Löf type theory:

- **Directed type theory** [LH11; Nuy15; Nor19; AN24]: replace identity types $x =_A y$ by asymmetric *hom types* $\mathrm{hom}_A(x, y)$.
- **Simplicial type theory** [RS17; BW23; GWB24]: extend MLTT with new type layers that allow "probing" types, leading to categorical structure (Segal/Rezk types).

Our approach is **radically different**:

- Starting point: language + axioms from [CCNW];
- Find a type theory making these rigorous.

Approach of [CCNW]

Three-step process:

- Formulate axioms in model-agnostic language,
- Build up theory,
- Pick “underlying substrate”.

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Choice of model-agnostic language:

- **(1) Primitive notions:**
  - Structure: $(\infty, 1)$-categories, functors, natural isos, ...
  - Coherences: Composition, identities, associativity, ...
- **(2) Basic constructors:**
  - Terminal/initial $(\infty, 1)$-category;
  - (Co)products;
  - Pullbacks;
  - Functor categories.

(1) Primitives

Structure:

- $(\infty, 1)$-categories $C, D, E, \ldots;$
- Given C,D, functors $F, G \colon C \to D$;
- Given F, G, natural isomorphisms $\alpha, \beta \colon F \xrightarrow{\sim} G$;
- Given α, β, 3-isos $H \colon \alpha \xrightarrow{\sim} \beta$;
- Etcetera

Coherences:

- Composition of functors/natural isos/...
- Unitality, associativity, inverses
- Whiskering, horizontal composition, ...

Ivan's insight: Use Grothendieck-Maltiniotis weak $(\omega, 1)$-categories!

Globular set:

- Set X_0 of objects ('0-cells')
- For $x, y \in X_0$, a set $X_1(x, y)$ of morphisms ('1-cells')
- For $f, g \in X_1(x, y)$, a set $X_2(f, g)$ of 2-cells
- Etcetera

Weak $(\omega, 1)$-categories: Globular set with:

- Operations (compositions, whiskerings, ...);
- Coherences (unitality, associativity, ...)

Formulated using Grohendieck's "coherators".
Reformulated using type theories GSeTT/CaTT (Finster-Mimram) [FM17].

Type theory GSeTT

Judgments:

$$\Gamma \vdash, \qquad \Gamma \vdash A, \qquad \Delta \vdash \gamma : \Gamma, \qquad \Gamma \vdash t : A.$$

Inference rules:

$$\frac{}{\emptyset \vdash} \qquad \frac{\Gamma \vdash \qquad \Gamma \vdash A \qquad x \notin \mathrm{Var}(\Gamma)}{\Gamma, x : A \vdash}$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \star} \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \to_A u}$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \langle\rangle : \emptyset} \qquad \frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma, x : A \vdash \qquad \Delta \vdash t : A[\gamma]}{\Delta \vdash \langle \gamma, x \mapsto t \rangle : (\Gamma, x : A)}$$

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A}$$

Pasting contexts

Question: When should we get coherences?

$$x \xrightarrow{f} y \xrightarrow{g} z, \qquad x \underset{f'}{\overset{f}{\rightrightarrows}}\!\!{\scriptstyle \Downarrow\alpha}\; y \xrightarrow{\quad g \quad} z\ , \qquad x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$$

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Answer: Pasting contexts! Generated by two rules:

- Base pasting context: $(x : \star)$
- Context extension at 'dangling variable': $\Gamma, y : A, f : x \to_A y$.

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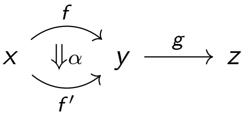

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Rules:

$$\frac{}{(x : \star) \vdash_{\mathrm{ps}} x : \star} \qquad \frac{\Gamma \vdash_{\mathrm{ps}} f : x \to_A y}{\Gamma \vdash_{\mathrm{ps}} y : A} \qquad \frac{\Gamma \vdash_{\mathrm{ps}} x : A}{\Gamma \vdash_{\mathrm{ps}}}$$

$$\frac{\Gamma \vdash_{\mathrm{ps}} x : A}{\Gamma, y : A, f : x \to_A y \vdash_{\mathrm{ps}} f : x \to_A y} \quad \text{when } y, f \notin \mathrm{Var}(\Gamma).$$

Coherences

Idea: Every pasting context has a unique "total composite".

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash u : A \qquad \Gamma \vdash v : A}{\Gamma \vdash \mathsf{coh}_{\Gamma, u \to_A v} : u \to_A v}\,(\mathsf{COH})$$

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The side condition (COH) has two cases:

$$(\textsc{comp}) \quad \begin{cases} \mathrm{Var}(u : A) = \mathrm{Var}(\partial^- \Gamma) \\ \mathrm{Var}(v : A) = \mathrm{Var}(\partial^+ \Gamma) \end{cases}$$

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Read: "u and v are (higher coherences between) total composites of Γ"

Example coherences

Side conditions:

$$(\textsc{comp}) \quad \begin{cases} \mathrm{Var}(u : A) = \mathrm{Var}(\partial^- \Gamma) \\ \mathrm{Var}(v : A) = \mathrm{Var}(\partial^+ \Gamma) \end{cases} \qquad (\textsc{inv}) \quad \begin{cases} \mathrm{Var}(u : A) = \mathrm{Var}(\Gamma) \\ \mathrm{Var}(v : A) = \mathrm{Var}(\Gamma) \end{cases}$$

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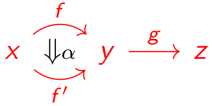
Examples (COMP):

$$x \xrightarrow{f} y \xrightarrow{g} z$$

$$u = x, \quad v = z$$

$$\rightsquigarrow \quad g \circ f \quad \text{(composition)}$$



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$$x \underset{f'}{\overset{f}{\rightrightarrows}} y \xrightarrow{g} z \qquad (\Downarrow \alpha : f \Rightarrow f')$$

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Example (INV):

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$$

$$u = h \circ (g \circ f), \quad v = (h \circ g) \circ f$$

$$\rightsquigarrow \quad \mathrm{assoc}_{h,g,f} \quad \text{(associator)}$$

From quasicategories to weak $(\omega, 1)$-categories

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Theorem (C.-Kobe)

Every quasicategory defines a weak $(\omega, 1)$*-category.*
In particular, the quasicategory $\mathrm{Cat}_{(\infty,1)}$ *of (small)* $(\infty, 1)$*-categories defines an* $(\omega, 1)$*-category.*

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Next up: Formulate basic axioms in CaTT.

(2) Basic constructors

Idea: Define constructors via universal properties:

- An object x is *terminal* if for every y there is a *unique* $y \to x$;
- A *product* $x \times y$ of x and y comes with $\mathrm{pr}_1 \colon x \times y \to x$ and $\mathrm{pr}_2 \colon x \times y \to y$, such that for $f \colon z \to x$ and $g \colon z \to y$ there is a *unique* $(f, g) \colon z \to x \times y$ s.t. $f \cong \mathrm{pr}_1 \circ (f, g)$ and $g \cong \mathrm{pr}_2 \circ (f, g)$.
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Uniqueness of a term $x : A$ in CaTT means:

- Existence of such $x : A$;
- For every two $x, y : A$ a unique term $\alpha : x \to_A y$.

(Semantically: the type A is 'weakly contractible').

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Iterated hom types of a type:
The judgment $\partial^*(B) \equiv A$ ("B is an iterated hom type of A") is defined by:

$$\frac{\Gamma \vdash A}{\Gamma \vdash \partial^*(A) \equiv A} \qquad\qquad \frac{\Gamma \vdash \partial^*(B) \equiv A \qquad \Gamma \vdash t, u : B}{\Gamma \vdash \partial^*(t \to_B u) \equiv A}$$

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The rules:

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- Have $!_{x \to 1} : x \to 1$;
- For $f, g : x \to 1$, have $!_{f \to g} : f \to g$ and $!_{g \to f} : g \to f$;
- Have $\mathrm{id}_f \to (!_{g \to f}) \circ (!_{f \to g})$, etcetera...

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- Etcetera...?

Higher whiskering:

Given $g : y \to_B z$, we have:

- For $f : x \to_B y$ a composite $g \circ f : x \to_B z$;
- For $\alpha : f \to_{x \to y} f'$ a whiskering $g \star \alpha : (g \circ f) \to_{x \to z} (g \circ f')$
- ...
- For $\alpha : A$ with $\partial^*(A) \equiv (x \to_B y)$, a *higher whiskering* $g \star \alpha : g \star A$.

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- Etcetera...?

The rules:

$$\frac{\Gamma \vdash x : \star \qquad \Gamma \vdash y : \star}{\Gamma \vdash x \times y : \star}$$

$$\frac{\begin{array}{l}\Gamma \vdash \partial^*(A) \equiv (z \to_\star x \times y) \\ \Gamma \vdash f : \mathrm{pr}_1 \star A \\ \Gamma \vdash g : \mathrm{pr}_2 \star A\end{array}}{\begin{array}{l}\Gamma \vdash (f, g) : A \\ \Gamma \vdash \mathrm{coh}_1^\times(f, g) : f \xrightarrow{\sim} \mathrm{pr}_1 \star (f, g) \\ \Gamma \vdash \mathrm{coh}_2^\times(f, g) : g \xrightarrow{\sim} \mathrm{pr}_2 \star (f, g)\end{array}}$$

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Details omitted.

Internal homs $[x, y] : \star$

Idea: Comes with $\mathrm{ev}\colon [x, y] \times x \to y$. For $f\colon z \to [x, y]$, obtain its *uncurrying* f^u*:*

$$z \times x \xrightarrow{f \times \mathrm{id}_x} [x, y] \times x \xrightarrow{\mathrm{ev}} y.$$

Universal property: for every $g\colon z \times x \to y$, there is a *unique currying* $g_c\colon z \to [x, y]$ with $(g_c)^u \cong g$.

Internal homs $[x, y] : \star$

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Universal property: for every $g\colon z \times x \to y$, there is a *unique currying* $g_c\colon z \to [x, y]$ with $(g_c)^u \cong g$.

The rules:

$$\frac{\vdash x : \star \qquad \vdash y : \star}{\begin{array}{l}\vdash [x, y] : \star \\ \vdash \mathrm{ev} : [x, y] \times x \to y\end{array}}$$

$$\frac{\begin{array}{l}\vdash \partial^*(A) \equiv (z \to_\star [x, y]) \\ \vdash g : \mathrm{ev} \star (A \times x)\end{array}}{\begin{array}{l}\vdash g_c : A \\ \vdash \mathrm{coh}^{[x,y]}(g) : g \xrightarrow{\sim} \mathrm{ev} \star (g_c \times x)\end{array}}$$

Conclusion

Upshot:
Type theory for $(\infty, 1)$-categories with products/pullbacks/internal homs.

Future: More synthetic category theory:

- Categories $[1] : \star$ and $[2] : \star$ encoding morphisms/commutative triangles;
- Segal/Rezk conditions;
- Groupoid cores: given $X \to C$, X groupoid, get unique $X \to C^{\simeq}$;
- Other category constructors, formulated via $\mathrm{Map}(C, D) := \mathrm{Fun}(C, D)^{\simeq}$.

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