Discrete differential geometry in homotopy type theory

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Summary

Summary

We make some constructions in HoTT that deserve to be called **geometry**:

- connections, curvature, and vector fields
- the index of a vector field
- a theorem in dimension 2 that total curvature = total index

Classical → HoTT

Let M be a compact, smooth, oriented 2-manifold without boundary, F_A the curvature of a connection A on the tangent bundle, and X a vector field with isolated zeroes $x_1, \ldots, x_n$.

$$\frac{1}{2\pi}\int_M F_A = \sum_{i=1}^{n} \mathrm{index}_X(x_i) = \chi(M) \qquad \leftarrow \text{Gauss-Bonnet/Poincaré-Hopf}$$

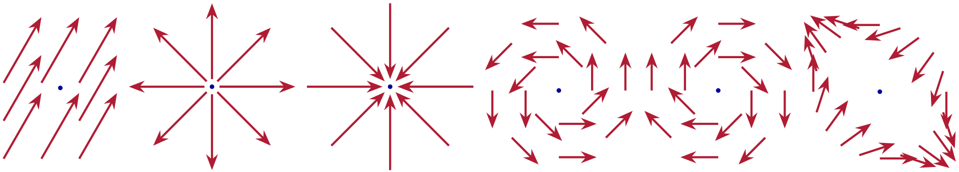
$$\downarrow \qquad\qquad \downarrow$$

$$\sum_{\text{faces } F} \flat_F = \sum_{\text{faces } F} L_F^X \qquad \leftarrow \text{our theorem}$$

Classical index

Near an isolated zero there are only three possibilities: index 0, 1, –1.

Index is the winding number of the field as you move clockwise around the zero.



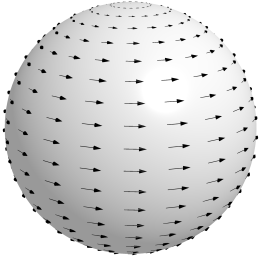
index 0 index +1 index +1 index +1 index +1 index –1

(This definition is using an implicit ad-hoc connection near the zero, more on this later.)

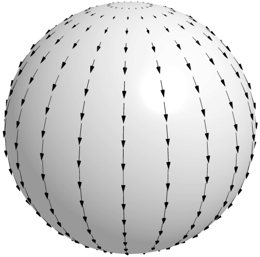
Poincaré-Hopf theorem

The total index of a vector field is the Euler characteristic.

Examples:



Rotation: index $+1$ at each pole $=$ **2**



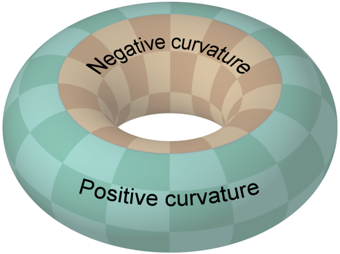
Height: index $+1$ at each pole $=$ **2**

Gauss-Bonnet theorem

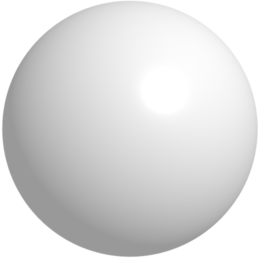
Total curvature divided by 2π is the Euler characteristic.

Curvature in 2D is a function $F_A : M \to \mathbb{R}$.

$\int_M F_A$ sums the values at every point.



Positive and negative curvature cancel: **0**



Constant curvature 1, area 4π: **2**

Plan

- Combinatorial manifolds
- Torsors and classifying maps
- Connections and curvature
- Vector fields
- Main theorem

HoTT background

1. **Symmetry**,
   Bezem, M., Buchholtz, U., Cagne, P., Dundas, B. I., and Grayson, D. R., (2021-)
   `https://github.com/UniMath/SymmetryBook`.
2. **Central H-spaces and banded types**,
   Buchholtz, U., Christensen, J. D. , Flaten, J. G. T., and Rijke, E. (2023)
   arXiv:2301.02636
3. **Nilpotent types and fracture squares in homotopy type theory**,
   Scoccola, L. (2020)
   MSCS 30(5). arXiv:1903.03245

Combinatorial manifolds

Manifolds in HoTT

- Recall the classical theory of **simplicial complexes**
- Define a **realization** procedure to construct types

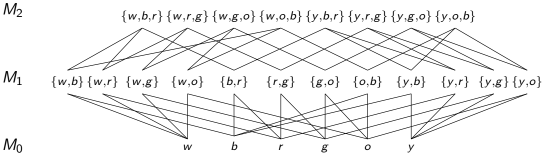
Simplicial complexes

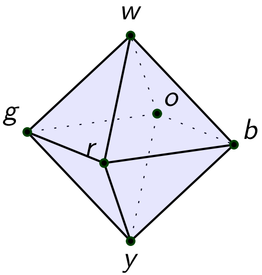
Definition

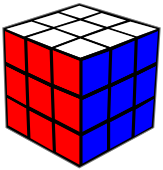
An **abstract simplicial complex** M **of dimension** n is an ordered list of sets $M \stackrel{\text{def}}{=} [M_0, \ldots, M_n]$ consisting of

- a set M_0 of vertices
- sets M_k of k-**faces**: subsets of M_0 of cardinality $k+1$
- downward closed: if $F \in M_k$ and $G \subseteq F$, $|G| = j+1$ then $G \in M_j$

We call the truncated list $M_{\leq k} \stackrel{\text{def}}{=} [M_0, \ldots, M_k]$ **the** k**-skeleton of** M.



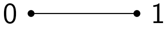




Simplicial complexes

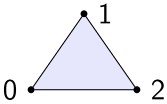
Example

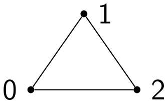
The **complete simplex of dimension** n, denoted $\Delta(n)$, is the set $\{0, \dots, n\}$ and its power set. The $(n-1)$-skeleton $\Delta(n)_{\leq(n-1)}$ is denoted $\partial\Delta(n)$ and will serve as a combinatorial $(n-1)$-sphere.

$\Delta(1)$ is visually 

, $\partial\Delta(1)$ is visually 

,

$\Delta(2)$ is visually 

, $\partial\Delta(2)$ is visually 

Homotopy realization: dimension 0

We will **realize** simplicial complexes by means of **a sequence of pushouts**.

It is likely we are exactly applying the $\int$ modality, but we won't need that.

Base case: the realization $\mathbb{M}$ of a 0-dimensional complex M is M_0.

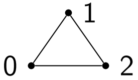
In particular the 0-sphere $\partial\mathbb{\Delta}(1) \stackrel{\text{def}}{=} \partial\Delta(1)_0$.

Homotopy realization: dimension 1

For a 1-dim complex $M \overset{\text{def}}{=} [M_0, M_1]$ the realization is given by

$$\begin{array}{ccc} M_1 \times \partial\Delta(1) & \xrightarrow{\mathsf{pr}_1} & M_1 \\ {\scriptstyle \mathbb{A}_0}\big\downarrow & \overset{h_1}{\Nearrow} & \big\downarrow{\scriptstyle *_{\mathbb{M}_1}} \\ M_0 = \mathbb{M}_0 & \longrightarrow & \ulcorner\,\mathbb{M}_1 \end{array}$$

Homotopy realization: dimension 1

For example the simplicial 1-sphere $\partial\Delta(2) \overset{\text{def}}{=}$ 

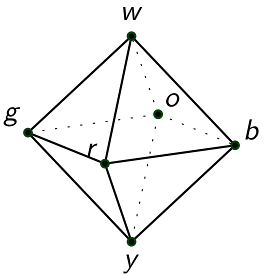
is given by

$$\begin{array}{ccc} \partial\Delta(2)_1 \times \partial\Delta(1) & \longrightarrow & \partial\Delta(2)_1 \\ \downarrow & \overset{h_1}{\Longrightarrow} \quad \lrcorner & \downarrow \\ \partial\Delta(2)_0 & \longrightarrow & \partial\Delta(2) \end{array} \quad \text{i.e.} \quad \begin{array}{ccc} \{\{0,1\},\{1,2\},\{2,0\}\} \times \{0,1\} & \longrightarrow & \{\{0,1\},\{1,2\},\{2,0\}\} \\ \downarrow & \overset{h_1}{\Longrightarrow} \quad \lrcorner & \downarrow \\ \{0,1,2\} & \longrightarrow & \partial\Delta(2) \end{array}$$

Homotopy realization: dimension 1

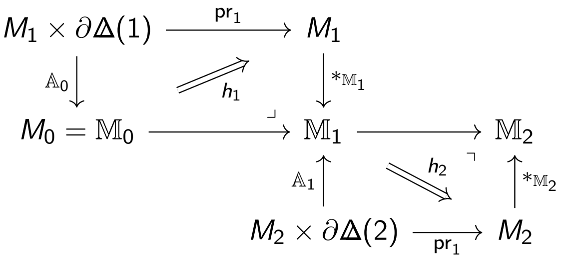
Or the 1-skeleton of the octahedron $\mathbb{O}$:

$$\begin{array}{ccc} \{\{w,g\},\ldots\} \times \{0,1\} & \longrightarrow & \{\{w,g\},\ldots\} \\ \downarrow & \overset{h_1}{\Nearrow} & \downarrow \\ \{w,g,\ldots\} & \longrightarrow & \mathbb{O}_1 \end{array}$$



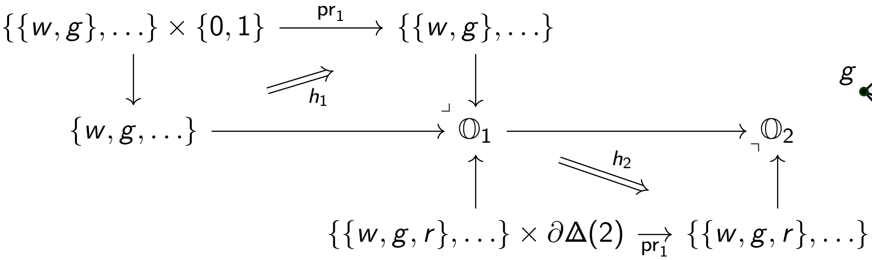
Homotopy realization: dimension 2

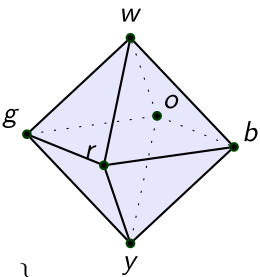
To realize $M \overset{\text{def}}{=} [M_0, M_1, M_2]$ use $\partial\Delta(1), \partial\Delta(2)$:



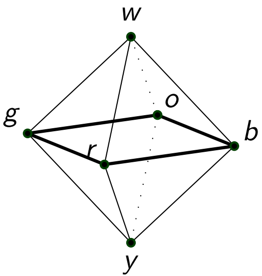
Homotopy realization: dimension 2

The full octahedron $\mathbb{O}$:





Homotopy realization: dimension 2



$\leftarrow \text{link}(w)$

The **link** of a vertex w in a 2-complex is: the faces not containing w but whose union with w is a face.

A **combinatorial** n**-manifold** is a simplicial complex all of whose links are* simplicial $(n-1)$-spheres.

Links will be our model of the **tangent space**.

*the (classical) geometric realization is homeomorphic to a sphere

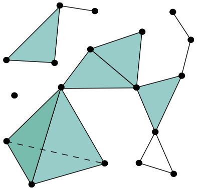
Combinatorial manifolds $\leftrightarrow$ smooth manifolds

Theorem (Whitehead (1940))

Every smooth n-manifold has a compatible structure of a **combinatorial manifold***: a simplicial complex of dimension n such that the link is a combinatorial* $(n-1)$*-sphere, i.e. its geometric realization is an* $(n-1)$*-sphere.*

https://ncatlab.org/nlab/show/triangulation+theorem

Counterexample: Wikipedia says this is a simplicial complex, but we can see it fails the link condition:



Torsors

What type families $\mathbb{M} \to \mathcal{U}$ will we consider? Families of **torsors**, also called **principal bundles**.

Torsors

Let G be a (higher) group.

Definition

- A **right** G**-object** is a type X equipped with a homomorphism $\phi : G^{\text{op}} \to \text{Aut}(X)$.
- X is furthermore a G**-torsor** if it is inhabited and the map $(\text{pr}_1, \phi) : X \times G \to X \times X$ is an equivalence.
- The inverse is (pr_1, s) where $s : X \times X \to G$ is called **subtraction** (when G is commutative).
- Let BG be the type of G-torsors.
- Let G_{reg} be the G-torsor consisting of G acting on itself on the right.

Facts we will import

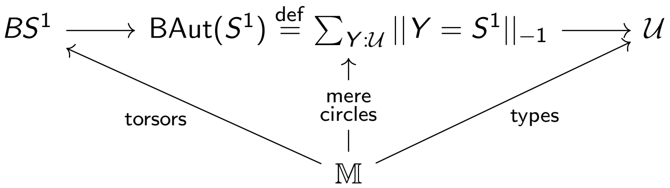
1. $\Omega(BG, G_{\text{reg}}) \simeq G$ and composition of loops corresponds to multiplication in G.
2. BG is connected.
3. 1 & 2 $\implies$ BG is a $\mathrm{K}(G, 1)$.

See the Buchholtz et al. H-spaces paper for more.

How to map into BS^1

To construct maps into BS^1 we **lift** a family of **mere circles**.

families of:



We will assume we have such a lift when we need it. (Remark: the lift is a choice of **orientation**.)

This point of view is explored in the Scoccola paper.

Other names:

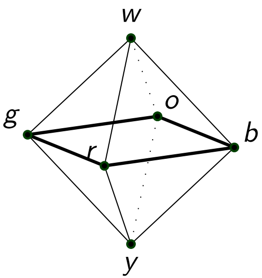
- $\mathrm{BAut}(S^1) = BO(2) = \mathrm{EM}(\mathbb{Z}, 1)$ (where $\mathrm{EM}(G, n) \overset{\text{def}}{=} \mathrm{BAut}(\mathrm{K}(G, n))$)
- $BS^1 = BSO(2) = \mathrm{K}(\mathbb{Z}, 2)$

Connections and curvature

Connections

Connections are extensions of a bundle to higher skeleta.

Recall link



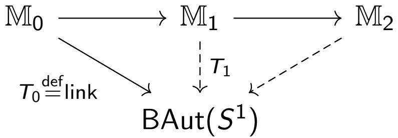
$\leftarrow \text{link}(w)$

The **link** of a vertex w in a 2-complex is: the faces not containing w but whose union with w is a face.

Define **the tangent bundle** on a combinatorial manifold to be
$T_0 \stackrel{\text{def}}{=} \text{link} : \mathbb{M}_0 \to \text{BAut}(S^1)$.

Connections on the tangent bundle

An extension T_1 of T_0 to $\mathbb{M}_1$ is called **a connection on the tangent bundle**.



What's the classical motivation for extending T to paths?

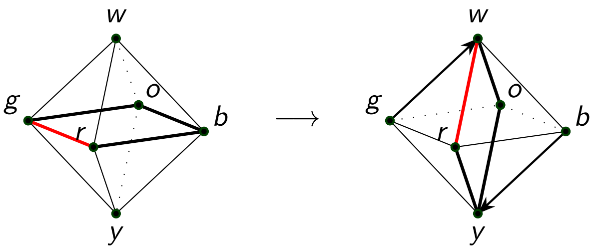
1. To differentiate sections of bundles.
2. To associate points in different fibers, which requires point 1 and integrating along a path.

(See **Gauge fields, knots and gravity**, Baez, J., Muniain, J. (1994).)

$T_1 : \mathbb{M}_1 \to \mathrm{BAut}(S^1)$ extending link

We will define T_1 on the edge wb, so we need a term $T_1(wb) : \mathrm{link}(w) =_{\mathrm{BAut}(S^1)} \mathrm{link}(b)$.

We imagine tipping:

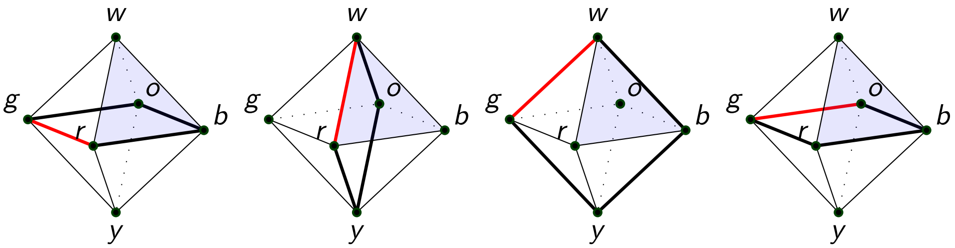


$T_1(g : \mathrm{link}(w)) \stackrel{\mathrm{def}}{=} w : \mathrm{link}(b), \ldots.$

Use this method to define T_1 on every edge.

$T_1 : \mathbb{M}_1 \to \mathrm{BAut}(S^1)$ extending link

Denote the path $wb \cdot br \cdot rw$ by $\partial(wbr)$. Consider $T_1(\partial(wbr))$:



We come back rotated by 1/4 turn. Call this rotation $R : \mathrm{link}(w) =_{BAut(S^1)} \mathrm{link}(w)$.

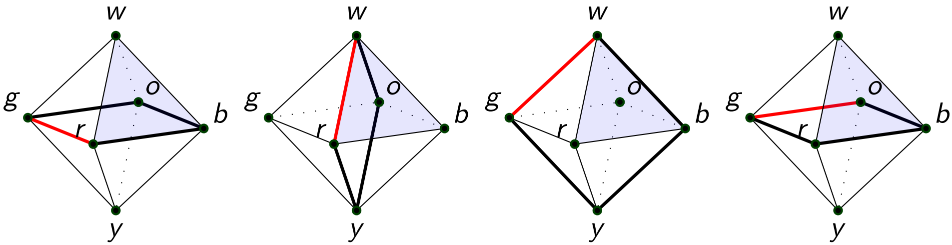
Extending T_1 to a face

Let $H_{wbr} : \mathrm{refl}_w =_{w =_{\mathbb{M}} w} \partial(wbr)$ be the filler homotopy of the face.

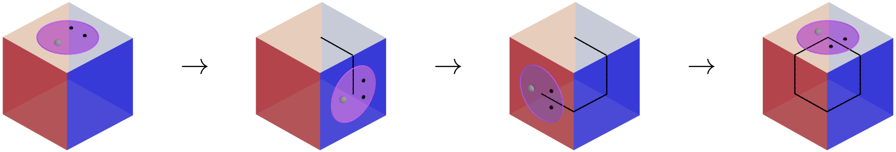
T_2 must live in $T_1(\mathrm{refl}_w) =_{(\mathrm{link}(w) =_{\mathrm{BAut}(S^1)} \mathrm{link}(w))} T_1(\partial(wbr)) = R$

T_2 must be a homotopy $H_R : \mathrm{id} = R$ between automorphisms of $\mathrm{link}(w)$.

For example, a path $H_R(g) : g = Rg = o$. Choose go.



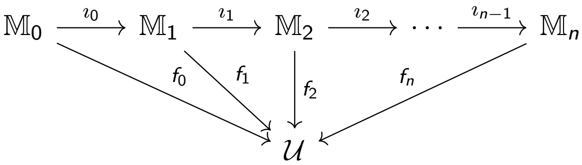
Original inspiration



The definition of a connection

Definition

If $\mathbb{M} \overset{\text{def}}{=} \mathbb{M}_0 \xrightarrow{\imath_0} \cdots \xrightarrow{\imath_{n-1}} \mathbb{M}_n$ is the realization of a combinatorial manifold and all the triangles commute in the diagram:

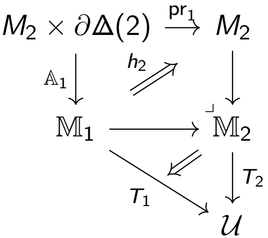


- The map f_k is a k-**bundle** on $\mathbb{M}$.
- The pair given by the map f_k and the proof $f_k \circ \imath_{k-1} = f_{k-1}$, i.e. that f_k extends f_{k-1} is called a k-**connection on the** $(k-1)$-**bundle** f_{k-1}.

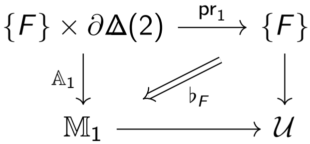
The definition of curvature

Definition (cont.)

An extension consists of M_2-many extensions to faces:



Here's the outer square for a single face F:



$T_1(\partial(F))$ is **the curvature at the face** F and the filler $\flat_F : \mathrm{id} = T_1(\partial F)$ is called a **flatness structure for the face** F.

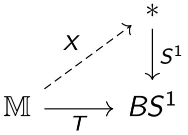
The distinction between the path $\flat_F$ and the endpoint $T_1(\partial(F))$ is small enough to be confusing.

Vector fields

Vector fields

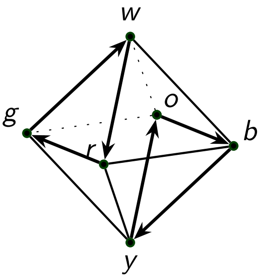
Let $T : \mathbb{M} \to BS^1$ be an oriented tangent bundle on a 2-dim realization of a combinatorial manifold.

- Our bundles of mere circles can only model **nonzero** tangent vectors.
- A global section of this family would be a trivialization of T, so that's not a good definition.

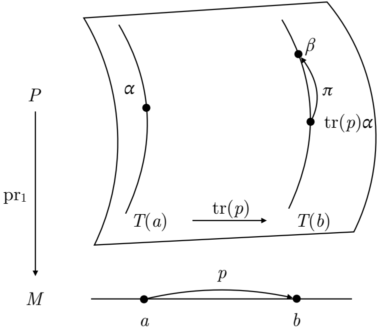


Our solution:

- A **vector field** is a term $X : \prod_{m:\mathbb{M}_1} Tm$.
- It models a classical **nonvanishing** vector field on the 1-skeleton.
- We model classical zeros by omitting the faces.

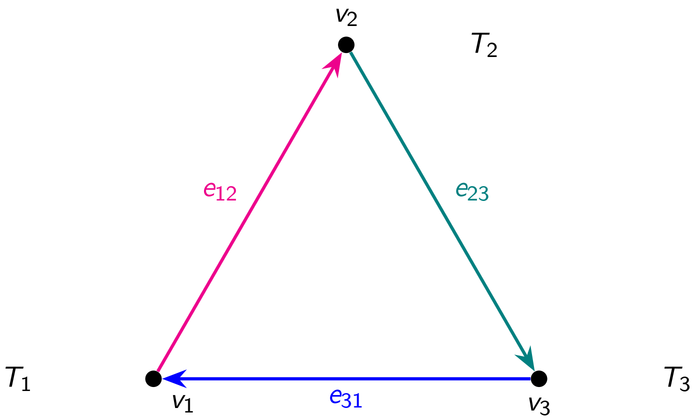


Reminder: pathovers

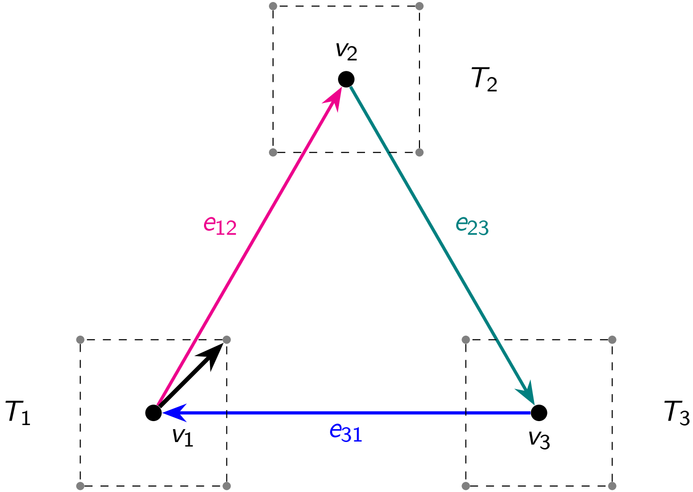


- Recall pathovers (dependent paths).
- There is an asymmetry: we pick a fiber to display π, the path over p.
- Dependent functions map paths to pathovers: $\mathsf{apd}(X)(p) : \mathsf{tr}_p(X(a)) = X(b)$ (simply denoted $X(p)$).

Next goal: define the index of a vector field on a face by computing $X(\partial F)$ around a face.



An example of **swirling** and **index** at this face.



An example of **swirling** and **index** at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.
-
-
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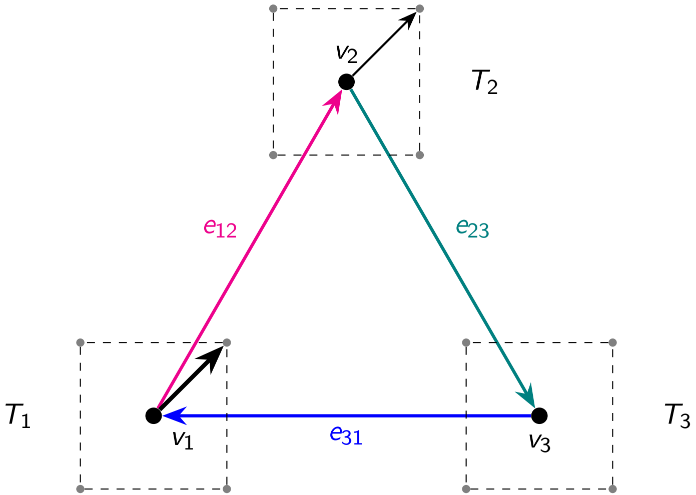

An example of **swirling** and **index** at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{21} is trivial. Denote the transported vector as thinner.
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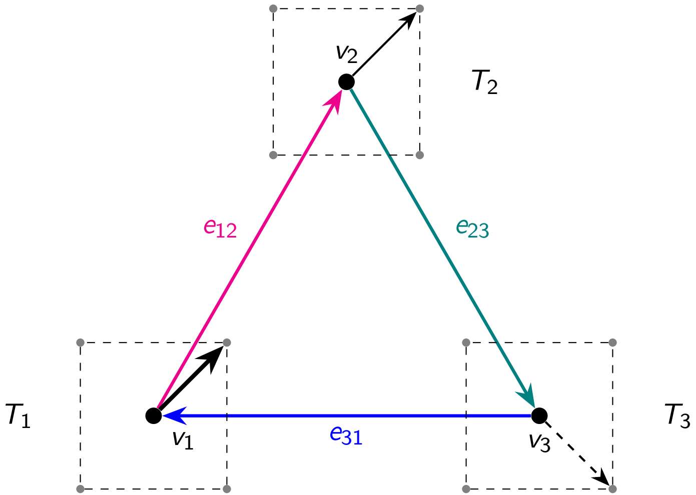

An example of **swirling** and **index** at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{21} is trivial. Denote the transported vector as thinner.
- Say T_{32} rotates clockwise. Denote the twice-transported vector as dashed.
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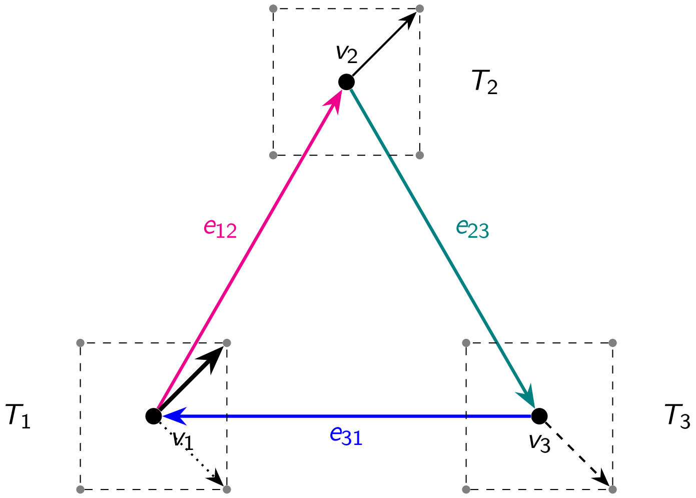
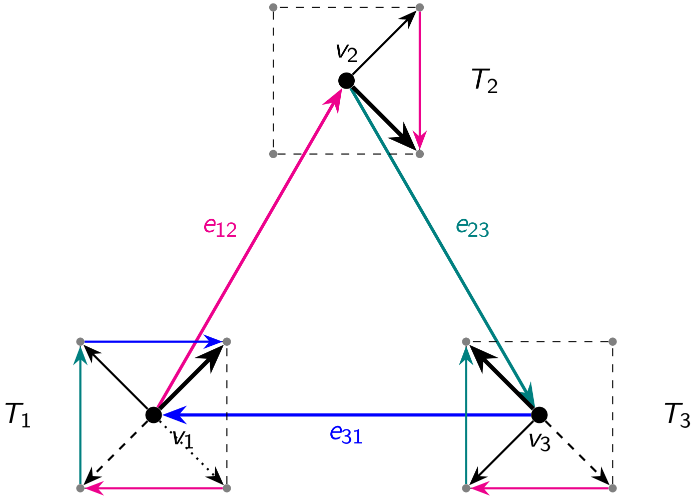


An example of **swirling** and **index** at this face.

- Denote by X_1 this vector $X(v_1) : T_1$.
- Say T_{21} is trivial. Denote the transported vector as thinner.
- Say T_{32} rotates clockwise. Denote the twice-transported vector as dashed.
- Say T_{13} is trivial. The thrice-transported vecor is dotted.



- X on e_{12} is red, etc.
- We translated all pathover data to the end of the loop.
- (Reminds me of scooping ice cream towards the last fiber.)
- The total pathover $X(\partial F)$ is called **the swirling** X_F of X at the face F.

Symbolic version

$$
\begin{array}{cccc}
T_1 \xrightarrow{T_{21}} & T_2 \xrightarrow{T_{32}} & T_3 \xrightarrow{T_{13}} & T_1 \\
 & & & T_{13}\,T_{32}\,T_{21}\,X_1 \\
 & & & T_{13}\,T_{32}X_{21} : \| \\
 & & T_{32}\,T_{21}\,X_1 & T_{13}\,T_{32}X_2 \\
 & & T_{32}X_{21} : \| & T_{13}X_{32} : \| \\
 & T_{21}X_1 & T_{32}X_2 & T_{13}X_3 \\
 & X_{21} : \| & X_{32} : \| & X_{13} : \| \\
X_1 & X_2 & X_3 & X_1
\end{array}
$$

Index

$$\mathrm{tr}_F \stackrel{\mathrm{def}}{=} \mathrm{tr}(\partial F) \quad : T_1 =_{BS^1} T_1 \qquad \textbf{curvature}$$

$$\flat_F \stackrel{\mathrm{def}}{=} \flat(\partial F) \quad : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F \qquad \textbf{flatness}$$

$$X_F \stackrel{\mathrm{def}}{=} X(\partial F) \quad : \mathrm{tr}_F(X_1) =_{T_1} X_1 \qquad \textbf{swirling}$$

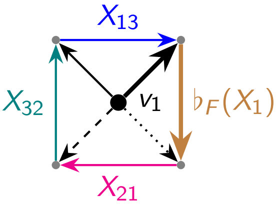
Definition

The **flattened swirling** of the vector field X on the face F is the loop

$$L_F^X \stackrel{\mathrm{def}}{=} \flat_F(X_1) \cdot X_F : X_1 =_{T_1} X_1.$$

The **index** of the vector field X on the face F is the integer I_F^X such that $\mathrm{loop}^{I_F^X} =_{S^1} \alpha(L_F^X)$ for some iso $\alpha : T_1 \simeq S^1$.

The classical index uses a trivial local connection in a chart. We use the given type family and then subtract the effect of curvature.



Main theorem

Towards totaling

"Total swirling" is some complicated pathover. But the vector field removes some dependency.

Pay off all our assumptions 1: torsor structure, vector field

$$
\begin{array}{c}
T_1 \\[2em]
T_{13}\,T_{32}\,T_{21}X_1 \\
T_{13}\,T_{32}X_{21} : \big\| \\
T_{13}\,T_{32}X_2 \\
T_{13}X_{32} : \big\| \\
T_{13}X_3 \\
X_{13} : \big\| \\
X_1
\end{array}
$$

- Def: $\alpha_i \overset{\text{def}}{=} s(-, X_i) : T_i \xrightarrow{\sim} S^1$ (**trivialization on 0-skeleton**).
- Def: $\rho_{ji} \overset{\text{def}}{=} \alpha_j(T_{ji}(X_i))$ is **the rotation of** T_{ji}.

$$
\begin{array}{ccc}
T_i & \xrightarrow{T_{ji}} & T_j \\
\text{base}\mapsto X_i \circlearrowleft\ \alpha_i \downarrow & & \downarrow \alpha_j \circlearrowright\ \text{base}\mapsto X_j \\
S^1 & \xrightarrow[(-)+\rho_{ji}]{} & S^1
\end{array}
$$

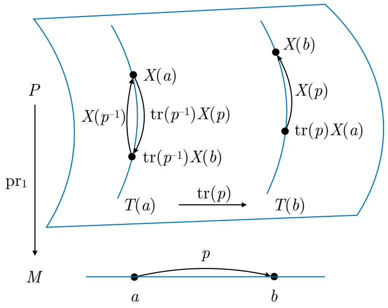
- Lemma: $\rho_{ij} = -\rho_{ji}$ because **in** T_j:
  $\rho_{ij} + \rho_{ji} + X_j = \rho_{ij} + T_{ji}X_i = T_{ji}(\rho_{ij} + X_i) = T_{ji}\,T_{ij}X_j = X_j$.

Pay off all our assumptions 1: torsor structure, vector field (cont.)

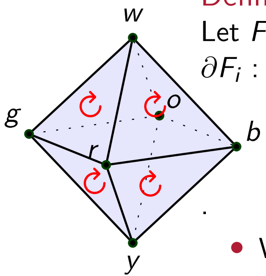
$$T_1$$

$$T_{13}T_{32}T_{21}X_1$$
$$T_{13}T_{32}X_{21}: \Vert$$
$$T_{13}T_{32}X_2$$
$$T_{13}X_{32}: \Vert$$
$$T_{13}X_3$$
$$X_{13}: \Vert$$
$$X_1$$

- Define $\sigma_{ji} \stackrel{\text{def}}{=} \alpha_j(X_{ji}) : \rho_{ji} =_{S^1} \text{base}$,. (compare $X_{ji} : T_{ji}X_i =_{T_j} X_j$)
- Paths of the form $(a =_{S^1} \text{base})$ can be added:
    - $+' : (a = \text{base}) \times (b = base) \to (a + b = base)$.
    - $p +' q = (p + b) \cdot q$.
- Lemma: $\sigma_{ij} +' \sigma_{ji} = \text{refl}_{\text{base}}$.
- Proof: $\text{apd}(X)(\text{refl}) = \text{refl}$ $\implies X_{ij} \cdot T_{ij}X_{ji} = \text{refl}_{X_i}$ $\implies \sigma_{ij} +' \sigma_{ji} = \text{refl}_{\text{base}}$ (T_{ij} just translates X_{ji} to cat with X_{ij}).



Pay off all our assumptions 2: no boundary, commutativity



Definition

Let $F_1, \ldots, F_n$ be the faces of $\mathbb{M}$, $v_i : F_i$ be designated vertices, and $\partial F_i : v_i = v_i$ be the triangular boundaries. The **total swirling** is

$$X_{\text{tot}} \overset{\text{def}}{=} \sigma_{\partial F_1} +' \cdots +' \sigma_{\partial F_n}$$

.

- We assume that this expression involves **every edge once in each direction**.
- S^1 is commutative, hence **complete cancellation**.

Consequence

$$\begin{aligned}
\mathrm{tr}_F &\overset{\text{def}}{=} \mathrm{tr}(\partial F) && : T_1 =_{BS^1} T_1 && \textbf{curvature} \\
\flat_F &\overset{\text{def}}{=} \flat(\partial F) && : \mathrm{id} =_{(T_1 =_{BS^1} T_1)} \mathrm{tr}_F && \textbf{flatness} \\
X_F &\overset{\text{def}}{=} X(\partial F) && : \mathrm{tr}_F(X_1) =_{T_1} X_1 && \textbf{swirling} \\
L_F^X &\overset{\text{def}}{=} \flat_F(X_1) \cdot X_F && : X_1 =_{T_1} X_1 && \textbf{flattened swirling}
\end{aligned}$$

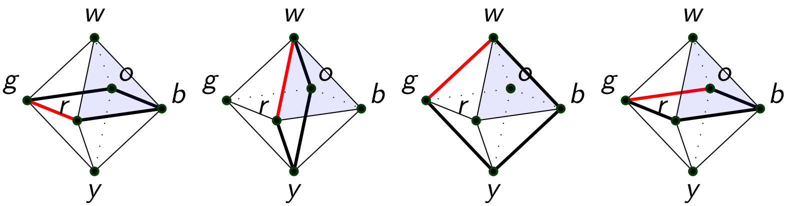
These can all be totaled in S^1 to give

$$\mathrm{tr}_{\mathrm{tot}} \overset{\text{def}}{=} \sum_i \rho_{\partial F} = \mathrm{base} \qquad X_{\mathrm{tot}} \overset{\text{def}}{=} \sum_i \sigma_{\partial F} = \mathrm{refl}_{\mathrm{base}}$$

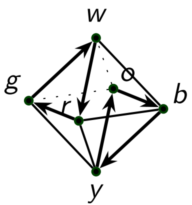
$$\flat_{\mathrm{tot}} \overset{\text{def}}{=} \sum_i \flat_{\partial F} \qquad L_{\mathrm{tot}}^X \overset{\text{def}}{=} \sum_i \flat_{\partial F} + \sigma_{\partial F} = \sum_i \flat_{\partial F}$$

So in our lingo: the total flatness equals the total flattened swirling. □

Examples

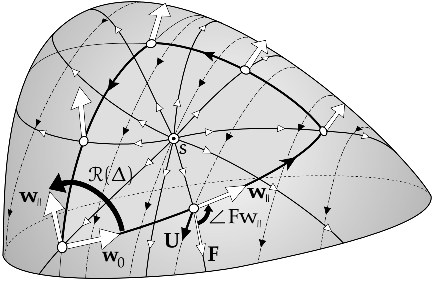


Each face contributes $\flat_F = H_R$, a 1/4-rotation. Total: 2.



For total index one obtains +1 from F_{wrg}, +1 from F_{ybo}, +0 from others. Total: 2.

Classical proof



[26.2] *The difference* $\mathcal{R}(\Delta) - 2\pi \mathcal{I}_F(s)$ *can be found by summing over the edges* K_j *the change* $\Phi(K_j)$ *in the illustrated angle* $\angle F w_{||}$, *i.e., the rotation of* $\mathbf{w}_{||}$ *relative to* $\mathbf{F}$.

Figure: Needham, T. (2021) Visual Differential Geometry and Forms.

- The classical proof is discrete-flavored.
- "$\angle Fw_{||}$" looked a lot like a pathover.
- Hopf's Φ is defined on edges, not loops. We imitated that too.

Thank you!

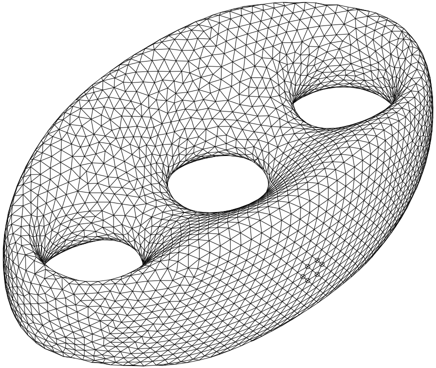
Appendix: the octahedron is too special

Not everything is a regular solid

The octahedron is very symmetrical. The links are all squares and the tippings rotate by 90 degrees.

A more typical situation has **variable** links, averaging to 6 sides. Also transport will rotate by **small** amounts.

Idea: introduce a **subdivider** D_n which takes a k-gon to an nk-gon, adding n vertices along each edge. Set $T_0(v) \stackrel{\text{def}}{=} D_{n(v)} \circ \text{link}(v)$. Make all tangent spaces the same LCM-sided polygon. Also transport should now be able to imitate small angles.



Appendix: Conjectural dictionary

Conjectionary

Homotopy realization likely amounts to the shape operator.

> *[G]iven a "cell complex" presentation of a classical topological space, if we can convert it into both a specification for a HIT and a colimit decomposition of that space that issufficiently "cofibrant", then $\int$ will preserve that colimit and take the space to the HIT.*
>
> — Mike Shulman, *Brouwer's Fixed Point Theorem in Real-Cohesive HoTT*

Conjectionary

Homotopy realization (and/or $\int$) may provide the following relationships.

Classical octahedron	Homotopy realization, e.g. $\mathbb{O}$
Classical combinatorial manifold M	$\int M$
Derivative	ap
Leibniz rule for $f, g : M \to \mathbb{R}$	Given H-space $(A, *)$, $f, g : X \to A$, $p : x =_X y$ then $\mathsf{ap}(f * g)(p) = \mathsf{ap}(f)(p) * (ga) \cdot (fb) * \mathsf{ap}(g)(p)$. Because in $A \times A$, $(fp, gp) = (fp, \mathsf{refl}) \cdot (\mathsf{refl}, gp)$
The sphere is not flat, as a **pointwise** statement	Nontrivial flatness on each face

Conjectionary

Connections being “affine”, and not (quite) 1-forms	$T_{ji} : T_i = T_j$ being a torsor and not (quite) a group
Space of connections for a given P is contractible.	Two extensions to $\mathbb{O}_1$...
Tensor	?

Conjectionary

Maurer-Cartan form.	Hmm, consider the trivial connection on $M \times G$ or $G \to *$.
Gauge transformations acting on connections and maybe functions (YM) of connections.	
The based gauge group acts freely on connections.	

Conjectionary

Characteristic classes.	$BS^1 \to B^n\mathbb{Z}$.
Chern-Weil theory.	$\mathbb{O} \xrightarrow{T} BS^1 \to B^n\mathbb{Z}$.
Hopf fibration.	$\mathbb{O} \xrightarrow{?} \mathrm{EM}(\mathbb{Z}, 1)$.
Zeros of X = Poincare dual of the Euler class.	