2-groups and Sính triples in homotopy type theory

HoTTEST

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Talk discusses joint work with Perry Hart

November 2026

Outline

1. Background
2. 2-groups in HoTT
3. Toolkit
4. A proof
5. Conclusion

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Per Buchholtz, Van Doorn and Rijke,[1] working in homotopy type theory, we have a couple of different equivalent ways to present the category of (1-)groups.

[1] U. Buchholtz, F. van Doorn, and E. Rijke. "Higher groups in homotopy type theory". In: *LICS '18: Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*. 2018.

Per Buchholtz, Van Doorn and Rijke,[1] working in homotopy type theory, we have a couple of different equivalent ways to present the category of (1-)groups.

An *algebraic* presentation

Objects are sets equipped with a binary operation, with a unit and inverses, morphisms are functions preserving this structure.

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An *algebraic* presentation

A *homotopical* presentation

Objects are pointed, connected 1-truncated spaces.

Morphisms are pointed maps of spaces.

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An *algebraic* presentation

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The equivalence sends an *algebraic* group $(G, \text{mult}, e, \text{inv})$ to its delooping BG. Which can be built as a HIT and satisfies a convenient universal property.

Example

The delooping of the integers is the circle, $B\,\mathbb{Z} = S^1$

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Conversely, a pointed, connected, 1-truncated space (X, x) is sent to its loop-space $(\Omega(X, x), \cdot, \mathtt{refl}, {}^{-1})$ considered as a group with path composition.

[1] Buchholtz, van Doorn, and Rijke, "Higher groups in homotopy type theory".

As Buchholtz, Van Doorn and Rijke[2] point out, the *homotopical* presentation of the category of 1-groups suggests a corresponding presentation of the category of n-groups.

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An n-group is a pointed, connected, n-truncated space.

Definition

A morphism of n-groups is a pointed map.

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Is there also an algebraic presentation of the category of n-groups?

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Classically the question has a positive answer.[3]

We're mostly going to focus on the case of 2-groups, but at the end of the slides I make some remarks about the general case.

[3] J.-L. Loday. "Spaces with finitely many non-trivial homotopy groups". In: *Journal of Pure and Applied Algebra* 24.2 (1982); M. Bullejos, A. M. Cegarra, and J. Duskin. "On cat^n-groups and homotopy types". In: *Journal of Pure and Applied Algebra* 86.2 (1993).

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Definition

A morphism of 2-groups is a pointed map.

Classically, 2-groups are very well understood, and there's a whole menagerie of positive answers to the question.[3]

[3] J. C. Baez and A. D. Lauda. "Higher-dimensional algebra V: 2-Groups". In: *Theory and Applications of Categories* 12.14 (2004).

Definition

A crossed module is a pair of groups (G, K) such that G acts on K, together with a homomorphism $\alpha : K \to G$, that satisfies equations.

$$\alpha(x) \cdot y = xyx^{-1}$$
$$\alpha(g \cdot x) = g\alpha(x)g^{-1}$$

The former is called the Peiffer identity.[4]

Example

If K is a group, then we can take $G = \mathrm{Aut}(K)$ and α sends an element of K to conjugation automorphism.

[4] R. Brown, P. J. Higgins, and R. Sivera. *Nonabelian algebraic topology*. European Mathematical Society, 2011.

Definition

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Definition

A monoidal groupoid is called an *algebraic* 2-group when every object x comes with an inverse x^{-1} and isomorphisms $\eta : I \to x \otimes x^{-1}$ and $\varepsilon : x^{-1} \otimes x \to I$ which satisfy the zig-zag identities.

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A Sính triple is a tuple (G, M, κ) where G is a group, M is an abelian group with an action by G, and κ is an element of the cohomology group $H^3(G; M)$.

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Theorem

Classically, *Sính triples are equivalent to crossed modules,*[4] *and to algebraic 2-groups.*[5]

[4] S. Mac Lane and J. H. C. Whitehead. "On the 3-type of a complex". In: *Proceedings of the National Academy of Sciences of the United States of America* 36 (1950).

[5] H. X. Sính. "Gr-catégories". PhD thesis. Institut Pédegagogique no. 2 de Hanoi, 1975.

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Von Branitz and Buchholtz[6] prove that the category of crossed modules is equivalent to the category of *strict algebraic* 2-groups **in HoTT**, where a strict algebraic 2-group is a monoidal category, equipped with inverses, whose space of objects is 0-truncated.

[6] J. Schipp von Branitz. "Higher groups via displayed univalent reflexive graphs in cubical type theory". MA thesis. Technische Universität Darmstadt, 2020; J. Schipp von Branitz and U. Buchholtz. "Using displayed univalent graphs to formalize higher groups in univalent foundations". 2021. URL: https://jsvb.xyz/files/durgs.pdf.

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They conjecture that the full equivalence for all algebraic 2-groups would follow from the *sets cover* axiom, which is independent of HoTT.

[6] Schipp von Branitz, "Higher groups via displayed univalent reflexive graphs in cubical type theory"; Schipp von Branitz and Buchholtz, "Using displayed univalent graphs to formalize higher groups in univalent foundations".

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Hart gave a proof that if G and G' are *univalent* algebraic 2-groups, then morphisms $G \to G'$ are equivalent to maps between their types of objects together with witnesses that the map preserves the monoidal product and the associator.

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Since every algebraic 2-group is equivalent to a univalent algebraic 2-group, this means we can define the HIT using a relatively small number of constructors.

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The equivalence sends an algebraic 2-group G to its delooping, $\mathcal{K}_2(G)$ built as a HIT.

It sends a homotopical 2-group to its loop-space, which is an algebraic 2-group where the monoidal structure comes from composition of paths between points, and the groupoid structure comes from composition of paths between paths.

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A write up of the proof, accompanying an agda formalization, appeared in the proceedings of LICS this year.[7]

[7] P. Hart and O. Milner. "Classifying 2-groups in homotopy type theory". In: *LICS 2026: Forty-First Annual Symposium on Logic in Computer Science*. Ed. by C. Faggian and J.-P. Katoen. 2026.

In the same conference paper we also wrote out a proof that the *type* of 2-groups is equivalent to the *type* of *synthetic* Sính triples **in HoTT**.

Definition

An synthetic Sính triple consists of a group G, a map $M : BG \to \mathrm{Ab}$, and a pointed dependent map $(c, \mathrm{pt}_c) : (x : BG) \to_{\mathrm{pt}} K(M(x), 3)$.

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Recalling that the group cohomology of G is equivalent to the cohomology of BG, and since Cavallo,[8] we define cohomology in HoTT as follows:

$$H^n(X; M) := \|(x : X) \to_{\mathrm{pt}} K(M(x), n)\|_0$$

when X is a space and M is any family of abelian groups over X.

[8] E. Cavallo. "Synthetic cohomology in homotopy type theory". MA thesis. Carnegie Mellon University, 2015.

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We suggested that a morphism of Sính triples should be defined as the following type of triples

$$(f : G \to_{\mathsf{Grp}} G') \times (g : (x : BG) \to M(x) \to_{\mathsf{Ab}} M'(Bf(x))) \times K(g(\bullet), 3) \circ c \sim_{\mathrm{pt}} c' \circ Bf$$

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In the rest of this talk, I'm going to show that, given two 2-groups, the type of morphisms between them is equivalent to this type of triples.

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We're trying to prove:

Theorem

If A and A' are two 2-groups, with corresponding synthetic Sính triples (G, M, c) and (G', M', c') then there is an equivalence of types

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\begin{aligned}
&(A \to_{\mathsf{pt}} A') \\
&\simeq \big((f : G \to_{\mathsf{Grp}} G') \times (g : (x : BG) \to M(x) \to_{\mathsf{Ab}} M'(Bf(x))) \\
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The last expression is the type of pointed homotopies

$$\begin{aligned}&\big(h : (x : BG) \to K(g(x), 3)(c(x)) = c'(Bf(x))\big)\\ &\times \mathrm{ap}_{K(g(\mathsf{pt}),3)}(\mathrm{pt}_c) =^{h(\mathsf{pt})} \mathrm{ap}_{c'}(\mathrm{pt}_{Bf}) \cdot \mathrm{pt}_{c'}\end{aligned}$$

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A map $M : BG \to \mathrm{Ab}$ is equivalently a single group $M(\mathrm{pt})$ with an action by G (using the universal property of BG)

The term g is equivalently a group homomorphism $g(\mathrm{pt}) : M(\mathrm{pt}) \to M'(Bf(\mathrm{pt}))$ which is compatible with the action

$$g(\alpha(x, y)) = \alpha'(x, g(y))$$

α and α' are the actions of G on $M(\mathrm{pt})$ and $M'(Bf(\mathrm{pt}))$.

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The next few slides will describe some of the tools we will use, and then we'll bring things together and conclude with the proof.

The first tool we'll need is a slightly more explicit description of the equivalence between 2-groups and Sính triples.

[8] D. Wärn. "Eilenberg-Maclane spaces and stabilisation in homotopy type theory". In: *Journal of Homotopy and Related Structures* 18 (2023).

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Inspecting the proof, which relied heavily on the methods of Wärn,[8] we discover that, if (G, M, c) is a Sính triple, the corresponding 2-group is

$$(x : BG) \times c(x) =_{K(M(x),3)} \mathrm{pt}$$

Pointed by $(\mathrm{pt}, \mathrm{pt}_c)$. A kind of "dependent fiber" of the map c.

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This implies that if (G, M, c) is the Sính triple corresponding to a homotopical 2-group A, then we have a pointed equivalence

$$A \simeq_{\mathrm{pt}} ((x : BG) \times c(x) = \mathrm{pt})$$

[8] Wärn, "Eilenberg-Maclane spaces and stabilisation in homotopy type theory".

The second tool we'll need comes from a paper of Scoccola.[9]

Scoccola considers the types $\mathtt{EM}(M, n)$ and $\mathtt{EM}_{\bullet,\bullet}(M, n)$, where the former are types merely equivalent to $K(M, n)$ and the latter are *bipointed* types merely equivalent to $(K(M, n), \mathrm{pt}, \mathrm{pt})$.

Scoccola constructs an isomorphism of types

$$R : \mathtt{EM}_{\bullet,\bullet}(M, n+1) \simeq \mathtt{EM}(M, n) : L$$

[9] L. Scoccola. "Nilpotent types and fracture squares in homotopy type theory". In: *Mathematical Structures in Computer Science* 30 (2020).

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Furthermore, there are equivalences

$$R' : (X \to_{\bullet,\bullet} Y) \simeq (LX \to LY)$$
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The equivalence R takes (X, x, x') to $x = x'$ and R' takes (f, p_1, p_2) to $\lambda x. p_1^{-1} \cdot \mathrm{ap}_f(x) \cdot p_2$.

[9] Scoccola, "Nilpotent types and fracture squares in homotopy type theory".

The final tool is just the universal property of Eilenberg-Mac Lane spaces.

There is an equivalence

$$K(\bullet, 3) : (M \to_{\mathsf{Ab}} M') \simeq (K(M, 3) \to_{\mathsf{pt}} K(M', 3))$$

which sends a homomorphism f to $K(f, 3)$, pointed by `refl`.

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With all that in place, let A and A' be two 2-groups (defined *homotopically*), let (G, M, c) and (G', M', c') be the corresponding untruncated Sính triples, and we can calculate:

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&(A \to_{\mathrm{pt}} A') \\
&\simeq (((x : BG) \times c(x) = \mathrm{pt}) \to_{\mathrm{pt}} ((x : BG') \times c'(x) = \mathrm{pt})) \\
&\simeq (f : (x : BG) \to c(x) = \mathrm{pt} \to ((x : BG') \times c'(x) = \mathrm{pt})) \\
&\quad \times (f(\mathrm{pt})(\mathrm{pt}_c) = (\mathrm{pt}, \mathrm{pt}_{c'})) \\
&\simeq (f : (x : BG) \to c(x) = \mathrm{pt} \to BG') \\
&\quad \times \ (g : (x : BG) \to (p : c(x) = \mathrm{pt}\ ) \to c'(f(x)(p)) = \mathrm{pt}) \\
&\quad \times (p_f : f(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) =^{p_f} \mathrm{pt}_{c'}) \\
&\simeq (f : BG \to BG') \\
&\quad \times (g : (x : BG) \to c(x) = \mathrm{pt} \to c'(f(x)) = \mathrm{pt}) \\
&\quad \times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'})
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&\simeq (((x : BG) \times c(x) = \mathrm{pt}) \to_{\mathrm{pt}} ((x : BG') \times c'(x) = \mathrm{pt})) \\
&\simeq (f : (x : BG) \to c(x) = \mathrm{pt} \to ((x : BG') \times c'(x) = \mathrm{pt})) \\
&\quad \times (f(\mathrm{pt})(\mathrm{pt}_c) = (\mathrm{pt}, \mathrm{pt}_{c'})) \\
&\simeq (f : (x : BG) \to c(x) = \mathrm{pt} \to BG') \\
&\quad \times \ (g : (x : BG) \to (p : c(x) = \mathrm{pt}\ ) \to c'(f(x)(p)) = \mathrm{pt}) \\
&\quad \times (p_f : f(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) =^{p_f} \mathrm{pt}_{c'}) \\
&\simeq (f : BG \to BG') \\
&\quad \times (g : (x : BG) \to c(x) = \mathrm{pt} \to c'(f(x)) = \mathrm{pt}) \\
&\quad \times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'})
\end{aligned}
$$

This was our first tool.

With all that in place, let A and A' be two 2-groups (defined *homotopically*), let (G, M, c) and (G', M', c') be the corresponding untruncated Sính triples, and we can calculate:

$$
\begin{aligned}
&(A \to_{\mathrm{pt}} A') \\
&\simeq (((x : BG) \times c(x) = \mathrm{pt}) \to_{\mathrm{pt}} ((x : BG') \times c'(x) = \mathrm{pt})) \\
&\simeq (f : (x : BG) \to c(x) = \mathrm{pt} \to ((x : BG') \times c'(x) = \mathrm{pt})) \\
&\quad \times (f(\mathrm{pt})(\mathrm{pt}_c) = (\mathrm{pt}, \mathrm{pt}_{c'})) \\
&\simeq (f : (x : BG) \to c(x) = \mathrm{pt} \to BG') \\
&\quad \times \ (g : (x : BG) \to (p : c(x) = \mathrm{pt}\ ) \to c'(f(x)(p)) = \mathrm{pt}) \\
&\quad \times (p_f : f(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) =^{p_f} \mathrm{pt}_{c'}) \\
&\simeq (f : BG \to BG') \\
&\quad \times (g : (x : BG) \to c(x) = \mathrm{pt} \to c'(f(x)) = \mathrm{pt}) \\
&\quad \times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'})
\end{aligned}
$$

By definition of the dependent sum.

With all that in place, let A and A' be two 2-groups (defined *homotopically*), let (G, M, c) and (G', M', c') be the corresponding untruncated Sính triples, and we can calculate:

$$
\begin{aligned}
&(A \to_{\mathrm{pt}} A') \\
&\simeq (((x : BG) \times c(x) = \mathrm{pt}) \to_{\mathrm{pt}} ((x : BG') \times c'(x) = \mathrm{pt})) \\
&\simeq (f : (x : BG) \to c(x) = \mathrm{pt} \to ((x : BG') \times c'(x) = \mathrm{pt})) \\
&\quad \times (f(\mathrm{pt})(\mathrm{pt}_c) = (\mathrm{pt}, \mathrm{pt}_{c'})) \\
&\simeq (f : (x : BG) \to c(x) = \mathrm{pt} \to BG') \\
&\quad \times \ (g : (x : BG) \to (p : c(x) = \mathrm{pt}\ ) \to c'(f(x)(p)) = \mathrm{pt}) \\
&\quad \times (p_f : f(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) =^{p_f} \mathrm{pt}_{c'}) \\
&\simeq (f : BG \to BG') \\
&\quad \times (g : (x : BG) \to c(x) = \mathrm{pt} \to c'(f(x)) = \mathrm{pt}) \\
&\quad \times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'})
\end{aligned}
$$

By the type theoretic axiom of choice.

With all that in place, let A and A' be two 2-groups (defined *homotopically*), let (G, M, c) and (G', M', c') be the corresponding untruncated Sính triples, and we can calculate:

$$
\begin{aligned}
&(A \to_{\mathrm{pt}} A') \\
&\simeq (((x : BG) \times c(x) = \mathrm{pt}) \to_{\mathrm{pt}} ((x : BG') \times c'(x) = \mathrm{pt})) \\
&\simeq (f : (x : BG) \to c(x) = \mathrm{pt} \to ((x : BG') \times c'(x) = \mathrm{pt})) \\
&\quad \times (f(\mathrm{pt})(\mathrm{pt}_c) = (\mathrm{pt}, \mathrm{pt}_{c'})) \\
&\simeq (f : (x : BG) \to c(x) = \mathrm{pt} \to BG') \\
&\quad \times \ (g : (x : BG) \to (p : c(x) = \mathrm{pt}\ ) \to c'(f(x)(p)) = \mathrm{pt}) \\
&\quad \times (p_f : f(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) =^{p_f} \mathrm{pt}_{c'}) \\
&\simeq (f : BG \to BG') \\
&\quad \times (g : (x : BG) \to c(x) = \mathrm{pt} \to c'(f(x)) = \mathrm{pt}) \\
&\quad \times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'})
\end{aligned}
$$

By interaction between dependent sums and identity types.

With all that in place, let A and A' be two 2-groups (defined *homotopically*), let (G, M, c) and (G', M', c') be the corresponding untruncated Sính triples, and we can calculate:

$$
\begin{aligned}
&(A \to_{\mathrm{pt}} A') \\
&\simeq (((x : BG) \times c(x) = \mathrm{pt}) \to_{\mathrm{pt}} ((x : BG') \times c'(x) = \mathrm{pt})) \\
&\simeq (f : (x : BG) \to c(x) = \mathrm{pt} \to ((x : BG') \times c'(x) = \mathrm{pt})) \\
&\quad \times (f(\mathrm{pt})(\mathrm{pt}_c) = (\mathrm{pt}, \mathrm{pt}_{c'})) \\
&\simeq (f : (x : BG) \to c(x) = \mathrm{pt} \to BG') \\
&\quad \times \ (g : (x : BG) \to (p : c(x) = \mathrm{pt}\ ) \to c'(f(x)(p)) = \mathrm{pt}) \\
&\quad \times (p_f : f(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) =^{p_f} \mathrm{pt}_{c'}) \\
&\simeq (f : BG \to BG') \\
&\quad \times (g : (x : BG) \to c(x) = \mathrm{pt} \to c'(f(x)) = \mathrm{pt}) \\
&\quad \times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'})
\end{aligned}
$$

Because $\mathrm{pt} = c(x)$ is 1-connected and BG' is 1-truncated.

With all that in place, let A and A' be two 2-groups (defined *homotopically*), let (G, M, c) and (G', M', c') be the corresponding untruncated Sính triples, and we can calculate:

$$
\begin{aligned}
&(A \to_{\mathrm{pt}} A') \\
&\simeq (((x : BG) \times c(x) = \mathrm{pt}) \to_{\mathrm{pt}} ((x : BG') \times c'(x) = \mathrm{pt})) \\
&\simeq (f : (x : BG) \to c(x) = \mathrm{pt} \to ((x : BG') \times c'(x) = \mathrm{pt})) \\
&\quad \times (f(\mathrm{pt})(\mathrm{pt}_c) = (\mathrm{pt}, \mathrm{pt}_{c'})) \\
&\simeq (f : (x : BG) \to c(x) = \mathrm{pt} \to BG') \\
&\quad \times \ (g : (x : BG) \to (p : c(x) = \mathrm{pt}\ ) \to c'(f(x)(p)) = \mathrm{pt}) \\
&\quad \times (p_f : f(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) =^{p_f} \mathrm{pt}_{c'}) \\
&\simeq (f : BG \to BG') \\
&\quad \times (g : (x : BG) \to c(x) = \mathrm{pt} \to c'(f(x)) = \mathrm{pt}) \\
&\quad \times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'})
\end{aligned}
$$

The J-rule tells us how to rewrite pathovers between paths.

Okay, then

$$
\begin{aligned}
& (f : BG \to BG') \\
& \times (g : (x : BG) \to c(x) = \mathrm{pt} \to c'(x) = \mathrm{pt}) \\
& \times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'}) \\
\simeq\ & (f : BG \to BG') \\
& \times\ (g : (x : BG) \to R(K(M(x), 3), c(x), \mathrm{pt}) \to R(K(M'(f(x)), 3), c'(x), \mathrm{pt})) \\
& \times\ (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'}) \\
\simeq\ & (f : BG \to BG') \\
& \times\ (g : (x : BG) \to ((K(M(x), 3), c(x), \mathrm{pt}) \to_{\bullet,\bullet} (K(M'(f(x)), 3), c'(x), \mathrm{pt}))) \\
& \times\ (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (R'g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'}) \\
\simeq\ & (f : BG \to BG') \\
& \times\ (g : (x : BG) \to K(M(x), 3) \to_{\mathrm{pt}} K(M'(f(x)), 3)) \\
& \times (h : (x : BG) \to g(x)(c(x)) = c'(f(x))) \\
& \times\ (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (R'(g(\mathrm{pt}), h(\mathrm{pt}))(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'})
\end{aligned}
$$

Okay, then

$$
\begin{aligned}
&(f : BG \to BG') \\
&\times (g : (x : BG) \to c(x) = \mathrm{pt} \to c'(x) = \mathrm{pt}) \\
&\times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'}) \\
\simeq\ &(f : BG \to BG') \\
&\times (g : (x : BG) \to R(K(M(x), 3), c(x), \mathrm{pt}) \to R(K(M'(f(x)), 3), c'(x), \mathrm{pt})) \\
&\times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'}) \\
\simeq\ &(f : BG \to BG') \\
&\times (g : (x : BG) \to ((K(M(x), 3), c(x), \mathrm{pt}) \to_{\bullet,\bullet} (K(M'(f(x)), 3), c'(x), \mathrm{pt}))) \\
&\times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (R'g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'}) \\
\simeq\ &(f : BG \to BG') \\
&\times (g : (x : BG) \to K(M(x), 3) \to_{\mathrm{pt}} K(M'(f(x)), 3)) \\
&\times (h : (x : BG) \to g(x)(c(x)) = c'(f(x))) \\
&\times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (R'(g(\mathrm{pt}), h(\mathrm{pt}))(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'})
\end{aligned}
$$

Using Scoccola's isomorphism.

Okay, then

$$
\begin{aligned}
&(f : BG \to BG') \\
&\times (g : (x : BG) \to c(x) = \mathrm{pt} \to c'(x) = \mathrm{pt}) \\
&\times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'}) \\
\simeq\ &(f : BG \to BG') \\
&\times (g : (x : BG) \to R(K(M(x), 3), c(x), \mathrm{pt}) \to R(K(M'(f(x)), 3), c'(x), \mathrm{pt})) \\
&\times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'}) \\
\simeq\ &(f : BG \to BG') \\
&\times (g : (x : BG) \to ((K(M(x), 3), c(x), \mathrm{pt}) \to_{\bullet,\bullet} (K(M'(f(x)), 3), c'(x), \mathrm{pt}))) \\
&\times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (R'g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'}) \\
\simeq\ &(f : BG \to BG') \\
&\times (g : (x : BG) \to K(M(x), 3) \to_{\mathrm{pt}} K(M'(f(x)), 3)) \\
&\times (h : (x : BG) \to g(x)(c(x)) = c'(f(x))) \\
&\times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (R'(g(\mathrm{pt}), h(\mathrm{pt}))(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'})
\end{aligned}
$$

Using the equivalence between morphism types from our second tool.

Okay, then

$$
\begin{aligned}
&(f : BG \to BG') \\
&\times (g : (x : BG) \to c(x) = \mathrm{pt} \to c'(x) = \mathrm{pt}) \\
&\times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'}) \\
\simeq\ &(f : BG \to BG') \\
&\times \ (g : (x : BG) \to R(K(M(x), 3), c(x), \mathrm{pt}) \to R(K(M'(f(x)), 3), c'(x), \mathrm{pt})) \\
&\times \ (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'}) \\
\simeq\ &(f : BG \to BG') \\
&\times \ (g : (x : BG) \to ((K(M(x), 3), c(x), \mathrm{pt}) \to_{\bullet,\bullet} (K(M'(f(x)), 3), c'(x), \mathrm{pt}))) \\
&\times \ (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (R'g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'}) \\
\simeq\ &(f : BG \to BG') \\
&\times \ (g : (x : BG) \to K(M(x), 3) \to_{\mathrm{pt}} K(M'(f(x)), 3)) \\
&\times (h : (x : BG) \to g(x)(c(x)) = c'(f(x))) \\
&\times \ (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (R'(g(\mathrm{pt}), h(\mathrm{pt}))(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'})
\end{aligned}
$$

A bipointed map is the same thing as a pointed map which preserves the second point...

Okay, then

$$
\begin{aligned}
&(f : BG \to BG') \\
&\times (g : (x : BG) \to c(x) = \mathrm{pt} \to c'(x) = \mathrm{pt}) \\
&\times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'}) \\
\simeq\ &(f : BG \to BG') \\
&\times \ (g : (x : BG) \to R(K(M(x), 3), c(x), \mathrm{pt}) \to R(K(M'(f(x)), 3), c'(x), \mathrm{pt})) \\
&\times \ (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'}) \\
\simeq\ &(f : BG \to BG') \\
&\times \ (g : (x : BG) \to ((K(M(x), 3), c(x), \mathrm{pt}) \to_{\bullet,\bullet} (K(M'(f(x)), 3), c'(x), \mathrm{pt}))) \\
&\times \ (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (R'g(\mathrm{pt})(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'}) \\
\simeq\ &(f : BG \to BG') \\
&\times \ (g : (x : BG) \to (M(x) \to_{\mathrm{Ab}} M'(f(x)))) \\
&\times (h : (x : BG) \to K(g(x), 3)(c(x)) = c'(f(x))) \\
&\times \ (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (R'(K(g(\mathrm{pt}), 3), h(\mathrm{pt}))(\mathrm{pt}_c) = \mathrm{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'})
\end{aligned}
$$

. . . then apply the third tool

We ended up with the term

$$
\begin{aligned}
&(f : BG \to BG') \\
&\times (g : (x : BG) \to (M(x) \to_{\mathsf{Ab}} M'(f(x)))) \\
&\times (h : (x : X) \to K(g(x), 3)(c(x)) = c'(f(x))) \\
&\times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (R'(K(g(\mathrm{pt}), 3), h(\mathrm{pt}))(\mathrm{pt}_c) = \mathrm{ap}_{c'}(\mathrm{pt}_f) \cdot \mathrm{pt}_{c'})
\end{aligned}
$$

but (f, p_f) is equivalent to the data of a group homomorphism $G \to_{\mathsf{Grp}} G'$, whose delooping will be f, h is a homotopy, and we just have to check that the last line gives a pointing for the homotopy.

We ended up with the term

$$
\begin{aligned}
&(f : BG \to BG') \\
&\times (g : (x : BG) \to (M(x) \to_{\mathsf{Ab}} M'(f(x)))) \\
&\times (h : (x : X) \to K(g(x), 3)(c(x)) = c'(f(x))) \\
&\times (p_f : f(\mathrm{pt}) = \mathrm{pt}) \times (R'(K(g(\mathrm{pt}), 3), h(\mathrm{pt}))(\mathrm{pt}_c) = \mathtt{ap}_{c'}(\mathrm{pt}_f) \cdot \mathrm{pt}_{c'})
\end{aligned}
$$

but (f, p_f) is equivalent to the data of a group homomorphism $G \to_{\mathsf{Grp}} G'$, whose delooping will be f, h is a homotopy, and we just have to check that the last line gives a pointing for the homotopy.

But it is an equation

$$h(\mathrm{pt})^{-1} \cdot \mathtt{ap}_{K(g(\mathrm{pt}),3)}(\mathrm{pt}_c) \cdot \mathtt{refl} = \mathtt{ap}_{c'}(p_f) \cdot \mathrm{pt}_{c'}$$

By unfolding the definition of R'.

Which gives us what we want after rewriting.

Outline

1 Background

2 2-groups in HoTT

3 Toolkit

4 A proof

5 Conclusion

In conclusion: we've discussed some progress on understanding the algebraic presentation of 2-groups.

We mentioned our earlier work showing that the type of 2-groups is equivalent to the type of Sính triples

$$(G : \mathrm{Grp}) \times (M : BG \to \mathrm{Ab}) \times ((x : BG) \to K(M(x), 3))$$

We presented a new proof, showing that a morphism of 2-groups $A \to_{\mathrm{pt}} A'$ is equivalent to a triple consisting of:

- A group homomorphism $f : G \to_{\mathrm{Grp}} G'$
- A group homomorphisms $g : M(\mathrm{pt}_{BG}) \to M'(f(\mathrm{pt}_{BG}))$ compatible with the actions (equivalently, the family of homomorphisms in the expression on previous slides).
- A pointed homotopy expressing that these group homomorphisms cohere with the cocycle in the Sính triple.

Theorem

Let $n \geq 1$, the type of $(n+1)$-groups is equivalent to the type of triples:

$$(G : n\,\mathrm{Grp}) \times (M : G \to \mathrm{Ab}) \times ((x : G) \to K(M(x), n+2))$$

Happily, our proof strategy above generalizes immediately, and we have the following:

Theorem

Let $n \geq 1$, if A and A' are two $(n+1)$-groups, with corresponding triples (G, M, c) and (G', M', c') then there is an equivalence of types

$$\begin{aligned}
&(A \to_{\mathrm{pt}} A') \\
&\simeq \big((f : G \to_{\mathrm{pt}} G') \times (g : (x : G) \to M(x) \to_{\mathrm{Ab}} M'(f(x))) \\
&\quad \times K(g(\bullet), n+2) \circ c \sim_{\mathrm{pt}} c' \circ f\big)
\end{aligned}$$

Question

Do these equivalences form a biequivalence between the bicategory of 2-groups and the bicategory of Sính triples?

This is work in progress.