Tiny Types and Cubical Type Theory

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Definition

A tiny object $\mathbb{T}$ in a category $\mathcal{C}$ is one for which $(\mathbb{T} \to -) : \mathcal{C} \to \mathcal{C}$ has a *right* adjoint $\surd : \mathcal{C} \to \mathcal{C}$.

- 1 in Set.
- The interval $\mathbb{I}$ in many versions of cubical sets.
- The infinitesimal disk $D :\equiv \{x : \mathbb{R} \mid x^2 = 0\}$ in models of synthetic differential geometry.
- The universal object in the topos classifying objects, [FinSet, Set].
- Any representable presheaf for a site with finite products.

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- The universal object in the topos classifying objects, $[\mathsf{FinSet}, \mathsf{Set}]$.
- Any representable presheaf for a site with finite products.

In a model of SDG, let $D :\equiv \{x : \mathbb{R} \mid x^2 = 0\}$.

The *tangent space* of X is the type $TX :\equiv D \to X$.

A (not-necessarily linear) *1-form* on X is a map

$$(D \to X) \to \mathbb{R}$$

By adjointness these are the same as maps

$$X \to \sqrt{\mathbb{R}}$$

Want a right adjoint to $(\mathbb{T} \to -)$ satisfying

- No axioms
- Allows $\mathbb{T}$ to be an ordinary type
- Comprehensible rules (relatively speaking)
- Usable by hand, informally and in a (hypothetical) proof assistant
- Plausible type-checking algorithm

- [LOPS18] axiomatises:

$$\begin{aligned} \surd &: \flat\mathcal{U} \to \mathcal{U} \\ \mathsf{R} &: \flat((\mathbb{T} \to A) \to B) \simeq \flat(A \to \surd B) \\ \mathsf{R}\text{-nat} &: \{R \text{ is natural in } A\} \end{aligned}$$

- [Mye22] improves to:

$$\begin{aligned} \surd &: \flat\mathcal{U} \to \mathcal{U} \\ \varepsilon &: (\mathbb{T} \to \surd B) \to B \\ \mathsf{e} &: \mathsf{isEquiv}(\flat(A \to \surd B) \to \flat((\mathbb{T} \to A) \to B)) \end{aligned}$$

Previous Approaches To Tininess

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Previous Approaches To Tininess

- [ND21; ND19; Nuy25] targets a right adjoint to "telescope quantification":

$$\frac{\Gamma, (\forall i : \mathbb{T}.\Delta) \vdash A \ \mathsf{type}}{\Gamma, i : \mathbb{T}, \Delta \vdash \between A \ \mathsf{type}}$$

- [GWB24] uses a system of MTT modalities together with an axiom $\Gamma, \{p\} \equiv \Gamma, i : \mathbb{T}$.

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The Less Amazing Right Adjoint

$$(-, x : A) \dashv (x : A) \to -$$

$$\frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda x.b : (x : A) \to B} \qquad \frac{\Gamma \vdash f : (x : A) \to B \qquad \Gamma \vdash a : A}{\Gamma \vdash f(a) : B[a/x]}$$

$$(\lambda x.b)(a) \equiv b[a/x] \qquad f \equiv \lambda x.f(x)$$

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$$\frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash \mathsf{lam}(b) : (x : A) \to B} \qquad \frac{\Gamma \vdash f : (x : A) \to B \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathsf{app}(f, a) : B[a/x]}$$

$$\mathsf{app}(\mathsf{lam}(b), a) \equiv b[a/x] \qquad f \equiv \mathsf{lam}(\mathsf{app}(f, x))$$

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$$\mathsf{unlam}(\mathsf{lam}(b)) \equiv b \qquad f \equiv \mathsf{lam}(\mathsf{unlam}(f))$$

The Fitch-Style Right Adjoint

$$\mathcal{L} \dashv \mathcal{R}$$

$$\frac{\Gamma, \mathcal{L} \vdash a : A}{\Gamma \vdash \mathsf{lam}(a) : \mathcal{R}A} \qquad \frac{\Gamma \vdash f : \mathcal{R}A}{\Gamma, \mathcal{L} \vdash \mathsf{unlam}(f) : A}$$

$$\mathsf{unlam}(\mathsf{lam}(b)) \equiv b \qquad f \equiv \mathsf{lam}(\mathsf{unlam}(f))$$

Following [BCMEPS20]. By $\Gamma, \mathcal{L}$ I mean $\mathcal{L}(\Gamma)$.

$$\mathcal{L} \dashv \mathcal{R}$$

$$\frac{\Gamma, \mathcal{L} \vdash a : A}{\Gamma \vdash \mathsf{lam}(a) : \mathcal{R}A} \qquad \frac{\Gamma \vdash f : \mathcal{R}A \qquad \mathcal{L} \notin \Gamma'}{\Gamma, \mathcal{L}, \Gamma' \vdash \mathsf{unlam}(f) : A}$$

$$\mathsf{unlam}(\mathsf{lam}(b)) \equiv b \qquad f \equiv \mathsf{lam}(\mathsf{unlam}(f))$$

Following [BCMEPS20]. By $\Gamma, \mathcal{L}$ I mean $\mathcal{L}(\Gamma)$.

$$\mathcal{E} \dashv \mathcal{L} \dashv \mathcal{R}$$

$$\frac{\Gamma, \mathcal{L} \vdash b : B}{\Gamma \vdash \mathsf{lam}(b) : \mathcal{R}B} \qquad \frac{\Gamma, \mathcal{E} \vdash f : \mathcal{R}B}{\Gamma, \mathcal{E}, \mathcal{L} \vdash \mathsf{unlam}(f) : B}$$

$$\mathsf{unlam}(\mathsf{lam}(b)) \equiv b \qquad f \equiv \mathsf{lam}(\mathsf{unlam}(f))$$

where

$$\textsc{unit}\ \frac{\Gamma, \mathcal{E}, \mathcal{L} \vdash \mathcal{J}}{\Gamma \vdash \mathcal{J}\{\eta\}} \qquad \textsc{counit}\ \frac{\Gamma \vdash \mathcal{J}}{\Gamma, \mathcal{L}, \mathcal{E} \vdash \mathcal{J}\{\varepsilon\}}$$

Following [GCKGB22].

$$\mathcal{E} \dashv \mathcal{L} \dashv \mathcal{R}$$

$$\frac{\Gamma, \mathcal{L} \vdash b : B}{\Gamma \vdash \mathsf{lam}(b) : \mathcal{R}B} \qquad \frac{\Gamma, \mathcal{E} \vdash f : \mathcal{R}B}{\Gamma \vdash \mathsf{app}(f) : B\{\eta\}}$$

$$\mathsf{app}(\mathsf{lam}(b)) \equiv b\{\eta\} \qquad f \equiv \mathsf{lam}(\mathsf{app}(f\{\varepsilon\}))$$

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$$(-, i : \mathbb{T}) \dashv (-, \text{🔒}) \dashv \surd$$

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Specialising to a tiny type.

$$(-, i : \mathbb{T}) \dashv (-, \text{🔒}) \dashv \surd$$

$$\frac{\Gamma, \text{🔒} \vdash b : B}{\Gamma \vdash \text{🔒}.b : \surd B} \qquad \frac{\Gamma, i : \mathbb{T} \vdash r : \surd B}{\Gamma \vdash r(\text{⅄}i.) : B\{\text{⅄}i./\text{🔒}\}}$$

$$(\text{🔒}.b)(\text{⅄}i.) \equiv b\{\text{⅄}i./\text{🔒}\} \qquad r \equiv \text{🔒}.r\{i/\text{🔑}\}(\text{⅄}i.)$$

where

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Specialising to a tiny type. $\text{⅄}i.$ binds i to its *left*.

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$$(🔒.b)(⅄ i.) \equiv b\{⅄ i./🔒\} \qquad r \equiv 🔒.r\{i/🔑\}(⅄ i.)$$

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Specialising to a tiny type. $⅄ i.$ binds i to its *left*.

$$(-, i : \mathbb{T}) \dashv (-, \text{🔒}_{\mathcal{L}}) \dashv \surd$$

$$\frac{\Gamma, \text{🔒}_{\mathcal{L}} \vdash b : B}{\Gamma \vdash \text{🔒}_{\mathcal{L}}.b : \surd_{\mathcal{L}} B} \qquad \frac{\Gamma, i : \mathbb{T} \vdash r : \surd_{\mathcal{L}} B}{\Gamma \vdash r(\text{⅄} i.) : B\{\text{⅄} i./\text{🔒}_{\mathcal{L}}\}}$$

$$(\text{🔒}_{\mathcal{L}}.b)(\text{⅄} i.) \equiv b\{\text{⅄} i./\text{🔒}_{\mathcal{L}}\} \qquad r \equiv \text{🔒}_{\mathcal{L}}.(r\{i/\text{🔑}_{\mathcal{L}}\}(\text{⅄} i.))$$

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Smoothing Out the Counit

$$\textsc{counit}\ \frac{\Gamma \vdash \mathcal{J}}{\Gamma, \text{🔒}, i : \mathbb{T} \vdash \mathcal{J}\{i/\text{🔑}\}}$$

Building in a substitution:

$$\textsc{counit}\ \frac{\Gamma \vdash \mathcal{J} \qquad \Gamma, \text{🔒} \vdash t : \mathbb{T}}{\Gamma, \text{🔒} \vdash \mathcal{J}\{t/\text{🔑}\}}$$

Building in some weakening:

$$\textsc{counit}\ \frac{\Gamma \vdash \mathcal{J} \qquad \Gamma, \text{🔒}, \Gamma' \vdash t : \mathbb{T} \qquad \text{🔒} \notin \Gamma'}{\Gamma, \text{🔒}, \Gamma' \vdash \mathcal{J}\{t/\text{🔑}\}}$$

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Definition

For closed* A, define $\mathsf{e} : \surd A \to A$ by

$$\mathsf{e}(r) :\equiv r(\curlyvee i.)$$

Compare:

$$\mathsf{const} : A \to (C \to A)$$
$$\mathsf{const}(a) :\equiv \lambda c.a$$

Example: Functoriality

$$\frac{\Gamma, \text{🔒} \vdash b : B}{\Gamma \vdash \text{🔒}.b : \sqrt{B}} \qquad \frac{\Gamma, i : \mathbb{T} \vdash r : \sqrt{B}}{\Gamma \vdash r(\text{⅄}i.) : B\{\text{⅄}i./\text{🔒}\}} \qquad \frac{\Gamma \vdash \mathcal{J} \qquad \Gamma, \text{🔒}, \Gamma' \vdash t : \mathbb{T}}{\Gamma, \text{🔒}, \Gamma' \vdash \mathcal{J}\{t/\text{🔑}\}}$$

Definition

For closed* $f : A \to B$, define $\sqrt{f} : \sqrt{A} \to \sqrt{B}$ by

$$(\sqrt{f})(r) :\equiv \text{🔒}.f(r\{i/\text{🔑}\}(\text{⅄}i.))$$

Given $r : \sqrt{A}$ we want $\sqrt{B}$. It suffices to produce B after locking our assumptions. Because we have $f : A \to B$ we just need an A. We don't have access to $r : \sqrt{A}$, because r is locked. We could unlock r as $r\{i/\text{🔑}\} : \sqrt{A}$ if only we had an assumption $i : \mathbb{T}$. Because we are eliminating $\sqrt{}$, we amazingly do have this assumption. So $(r\{i/\text{🔑}\})(\text{⅄}i.) : A$, and we can apply f.

Example: Functoriality

$$\frac{\Gamma, \text{🔒} \vdash b : B}{\Gamma \vdash \text{🔒}.b : \sqrt{B}} \qquad \frac{\Gamma, i : \mathbb{T} \vdash r : \sqrt{B}}{\Gamma \vdash r(\curlyvee i.) : B\{\curlyvee i./\text{🔒}\}} \qquad \frac{\Gamma \vdash \mathcal{J} \quad \Gamma, \text{🔒}, \Gamma' \vdash t : \mathbb{T}}{\Gamma, \text{🔒}, \Gamma' \vdash \mathcal{J}\{t/\text{🔑}\}}$$

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$$(\sqrt{f})(r) :\equiv \text{🔒}.f(r\{i/\text{🔑}\}(\curlyvee i.))$$

Compare:

$$f \circ - : (C \to A) \to (C \to B)$$
$$(f \circ -)(r) :\equiv \lambda c.f(r(c))$$

$$\frac{\Gamma, \text{🔒} \vdash b : B}{\Gamma \vdash \text{🔒}.b : \surd B} \qquad \frac{\Gamma, i : \mathbb{T} \vdash r : \surd B}{\Gamma \vdash r(\curlyvee i.) : B\{\curlyvee i./\text{🔒}\}} \qquad \frac{\begin{array}{c}\Gamma \vdash \mathcal{J} \\ \Gamma, \text{🔒}, \Gamma' \vdash t : \mathbb{T}\end{array}}{\Gamma, \text{🔒}, \Gamma' \vdash \mathcal{J}\{t/\text{🔑}\}}$$

Definition

For closed* $f : A \to B$, define $\surd f : \surd A \to \surd B$ by

$$(\surd f)(r, \text{🔒}) :\equiv f(r\{i/\text{🔑}\}(\curlyvee i.))$$

In an argument list, a 🔒 locks all variables to the left of it. When applied, the lock "argument" becomes a counit:

$$(\surd f)(s, \curlyvee i.) \equiv f(r\{i/\text{🔑}\}(\curlyvee i.))[s/r]\{\curlyvee i./\text{🔒}\}$$

Proposition

For types A and B, there is a map

$$\mathsf{unsplit} : (\mathbb{T} \to A + B) \to (\mathbb{T} \to A) + (\mathbb{T} \to B)$$

Proof.

$$\begin{aligned}
&\mathsf{lemma} : A + B \to \surd((\mathbb{T} \to A) + (\mathbb{T} \to B)) \\
&\mathsf{lemma}(\mathsf{inl}(a), \text{🔒}) :\equiv \mathsf{inl}(\lambda t.a\{t/\text{🔑}\}) \\
&\mathsf{lemma}(\mathsf{inr}(b), \text{🔒}) :\equiv \mathsf{inr}(\lambda t.b\{t/\text{🔑}\})
\end{aligned}$$

Then:

$$\mathsf{unsplit}(f) :\equiv \mathsf{lemma}(f(i), \text{⅄}i.)$$

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Then:

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□

The counit and unit commute with ordinary constructions.

$$(x, y)\{i/\key\} \equiv (x\{i/\key\}, y\{i/\key\})$$
$$(\lambda y.x + y)\{i/\key\} \equiv (\lambda y.x\{i/\key\} + y\{i/\key\})$$

$$(x, y)\{\text{⅄}i./\lock\} \equiv (x\{\text{⅄}i./\lock\}, y\{\text{⅄}i./\lock\})$$
$$(\lambda y.x + y)\{\text{⅄}i./\lock\} \equiv (\lambda y.x\{\text{⅄}i./\lock\} + y\{\text{⅄}i./\lock\})$$

When a unit meets a counit, it turns into a regular substitution:

$$\mathcal{J}\{t/\text{🔑}_{\mathcal{L}}\}\{\text{⅄}i./\text{🔒}_{\mathcal{L}}\} :\equiv \mathcal{J}[t/i]$$

In the simplest case,

$$\textsc{unit}\ \dfrac{\textsc{counit}\ \dfrac{\Gamma, i : \mathbb{T} \vdash \mathcal{J}}{\Gamma, i : \mathbb{T}, \text{🔒}_{\mathcal{L}} \vdash \mathcal{J}\{t/\text{🔑}_{\mathcal{L}}\}}}{\Gamma \vdash \mathcal{J}\{t/\text{🔑}_{\mathcal{L}}\}\{\text{⅄}i./\text{🔒}_{\mathcal{L}}\} \equiv \mathcal{J}[t/i]}$$

Or to make this more clearly a triangle identity:

$$\textsc{unit}\ \dfrac{\textsc{counit}\ \dfrac{\Gamma, i : \mathbb{T} \vdash \mathcal{J}}{\Gamma, i : \mathbb{T}, \text{🔒}_{\mathcal{L}}, j : \mathbb{T} \vdash \mathcal{J}\{j/\text{🔑}_{\mathcal{L}}\}}}{\Gamma, j : \mathbb{T} \vdash \mathcal{J}\{j/\text{🔑}_{\mathcal{L}}\}\{\text{⅄}i./\text{🔒}_{\mathcal{L}}\} \equiv \mathcal{J}[j/i]}$$

Counits are almost substitutions waiting to be "activated".

$$\begin{aligned} &f : \mathbb{T} \to \surd(\mathbb{T} \times \mathbb{T}) \\ &f(x, \text{\faLock}) :\equiv (x\{0/\text{\faKey}\}, x\{1/\text{\faKey}\}) \end{aligned}$$

(supposing some global elements $0, 1 : \mathbb{T}$)

$$\begin{aligned} &f(i, \yo i.) \\ &\equiv (i\{0/\text{\faKey}\}, i\{1/\text{\faKey}\})\{\yo i./\text{\faLock}\} \\ &\equiv (i\{0/\text{\faKey}\}\{\yo i./\text{\faLock}\}, i\{1/\text{\faKey}\}\{\yo i./\text{\faLock}\}) \\ &\equiv (i[0/i], i[1/i]) \equiv (0, 1) \end{aligned}$$

A single bound variable can have different things substituted for it in different places.

Paused Substitutions?

But not quite!

$$f : \mathbb{T} \to \surd \surd \mathbb{T}$$
$$f(x, \text{🔒}_{\mathcal{L}}, \text{🔒}_{\mathcal{K}}) :\equiv x\{0/\text{🔑}_{\mathcal{L}}\}\{1/\text{🔑}_{\mathcal{K}}\}$$

$$f(i, \mathsf{Y}i., \mathsf{Y}j.)$$
$$\equiv i\{0/\text{🔑}_{\mathcal{L}}\}\{1/\text{🔑}_{\mathcal{K}}\}\{\mathsf{Y}i./\text{🔒}_{\mathcal{L}}\}\{\mathsf{Y}j./\text{🔒}_{\mathcal{K}}\}$$
$$\equiv i[0/i][1/j] \equiv 0$$

$$f(j, \mathsf{Y}i., \mathsf{Y}j.)$$
$$\equiv j\{0/\text{🔑}_{\mathcal{L}}\}\{1/\text{🔑}_{\mathcal{K}}\}\{\mathsf{Y}i./\text{🔒}_{\mathcal{L}}\}\{\mathsf{Y}j./\text{🔒}_{\mathcal{K}}\}$$
$$\equiv j[0/i][1/j] \equiv 1$$

So the *user* of the term gets to choose which key is used.

[LOPS18] uses its version of $\sqrt{}$ to build an internal model of cubical type theory in intensional MLTT + Axioms.

I think we can do something a little different: use the present theory (+ a little more) to *implement* cubical type theory.

Fix a "notion of composition structure" $\mathsf{C} : (\mathbb{I} \to \mathcal{U}) \to \mathcal{U}$.

$$\mathsf{isFib} : (\Gamma : \mathcal{U}) \to (X : \Gamma \to \mathcal{U}) \to \mathcal{U}$$
$$\mathsf{isFib}(\Gamma, X) :\equiv (p : \mathbb{I} \to \Gamma) \to \mathsf{C}(X \circ p)$$

The [LOPS18] construction of a universe classifying (crisp) fibrations is:

$$\begin{array}{ccc} \mathcal{U}_{\mathsf{Fib}} & \longrightarrow & \surd(X : \mathcal{U}) \times X \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \surd \mathsf{pr}_1} \\ \mathcal{U} & \xrightarrow[\mathsf{C}^{\vee}]{} & \surd\mathcal{U} \end{array}$$

This classifies crisp fibrations in that, for $\Gamma :: \mathcal{U}$,

$$(\Gamma \to \mathcal{U}_{\mathsf{Fib}}) \cong ((X :: \Gamma \to \mathcal{U}) \times \mathsf{isFib}(\Gamma, X))$$

In our theory, the pullback works out to:

$$\mathcal{U}_{\mathsf{Fib}} :\equiv (X : \mathcal{U}) \times \surd \mathsf{C}(\lambda j. X\{j/\text{🔑}\})$$

So we're better off tweaking the definition of fibration

Definition

An *amazingly fibrant* type is a type X equipped with a term of

$$\mathsf{isAFib}(X) :\equiv \surd \mathsf{C}(\lambda j. X\{j/\text{🔑}\})$$

- A universe of judgemental propositions Cof closed under the same things as in CCTT.
- Cubical subtypes $A[\alpha \mapsto a_0]$ for $\alpha : \mathsf{Cof}$, roughly

$$A[\alpha \mapsto a_0] :\approx (a : A) \times ([\alpha] \to (a = a_0))$$

  so that magically $\mathsf{pr}_1(s) \equiv a_0$ when α holds.
- Glue types $\mathsf{Glue}(\alpha, B, T, f)$ for functions $\alpha \vdash f : T \to B$ into totally defined types B, roughly

$$\mathsf{Glue}(\alpha, B, T, f) :\approx (t : [\alpha] \to T) \times B[\alpha \mapsto f(t)]$$

  so that magically $\mathsf{Glue}(\alpha, B, T, f) \equiv T$ when α holds.
- *Not* coercion, composition, Path.

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- *Not* coercion, composition, Path.

The composition structure used in [ABCFHL21] is

$$\begin{aligned}\mathsf{C}(L) :\equiv{}& (\alpha : \mathsf{Cof}) \to (r : \mathbb{I}) \to (r' : \mathbb{I}) \\ &\to (P : (z : \mathbb{I}) \to [z = r \vee \alpha] \to L(z)) \\ &\to L(r')[r = r' \vee \alpha \mapsto P(r')]\end{aligned}$$

Plugging into the definition of amazing fibrancy:

$$\begin{aligned}\mathsf{isAFib}(X) :\equiv{}& \surd_{\mathcal{L}}(\alpha : \mathsf{Cof}) \to (r : \mathbb{I}) \to (r' : \mathbb{I}) \\ &\to (P : (z : \mathbb{I}) \to [z = r \vee \alpha] \to X\{z/\text{⚷}_{\mathcal{L}}\}) \\ &\to X\{r'/\text{⚷}_{\mathcal{L}}\}[r = r' \vee \alpha \mapsto P(r')]\end{aligned}$$

Being amazingly fibrant is *stronger* than the previous notion of fibrancy, so we have to re-check all the closure properties.

Suppose $A, B : \mathcal{U}$ with $\mathtt{comp}_A : \mathsf{isAFib}(A)$ and $\mathtt{comp}_B : \mathsf{isAFib}(B)$.

$$
\begin{aligned}
&\mathtt{comp}_{A\times B}(\text{🔒}, \alpha, r, r', t) \\
&:\equiv (\mathtt{comp}_A\{i/\text{🔑}\}(\text{⅄} i., \alpha, r, r', \lambda z.\mathsf{pr}_1(t(z))), \\
&\qquad \mathtt{comp}_B\{i/\text{🔑}\}(\text{⅄} i., \alpha, r, r', \lambda z.\mathsf{pr}_2(t(z))))
\end{aligned}
$$

Suppose $A : \mathcal{U}$ and $B : A \to \mathcal{U}$ with $\mathtt{comp}_A : \mathsf{isAFib}(A)$ and $\mathtt{comp}_B : (a : A) \to \mathsf{isAFib}(B(a))$.

$$
\begin{aligned}
&\mathtt{comp}_{(a:A)\times B(a)}(\text{🔒}, \alpha, r, r', t) \\
&:\equiv (\mathtt{comp}_A\{i/\text{🔑}\}(\text{Y}i., \alpha, r, r', \lambda z.\mathsf{pr}_1(t(z))), \\
&\qquad \mathtt{comp}_B\{i/\text{🔑}\}(a(i))(\text{Y}i., \alpha, r, r', \lambda z.\mathsf{pr}_2(t(z))))
\end{aligned}
$$

where

$$
a(i) :\equiv \mathtt{comp}_A\{j/\text{🔑}_{\mathcal{L}}\}(\text{Y}j., \alpha, r, i, \lambda z.\mathsf{pr}_1(t(z)))
$$

Implementation: Admissibility of Unit and Counit

We can push the counit and unit operations to the leaves.

- The counit gets stuck on variable uses, so needs to be built into the variable rule.

$$\textsc{var}\ \frac{\Gamma, x : A, \Gamma' \vdash \vec{t} : \mathbb{T} \qquad \text{for } \mathcal{L} \in \mathsf{locks}(\Gamma')}{\Gamma, x : A, \Gamma' \vdash x\{\!\{\vec{t}/\mathcal{L}\}\!\} : A\{\vec{t}/\text{🔑}_{\mathcal{L}}\}}$$

- The unit *never* gets stuck, and does not need any special treatment.

To have been used, encountered variables *must* have an attached key.

$$(x\{t/\text{🔑}\}, y\{t/\text{🔑}\})\{\text{⅄}i./\text{🔒}\} \equiv (x\{t/\text{🔑}\}\{\text{⅄}i./\text{🔒}\}, y\{t/\text{🔑}\}\{\text{⅄}i./\text{🔒}\})$$
$$(\lambda y.x\{t/\text{🔑}\} + y)\{\text{⅄}i./\text{🔒}\} \equiv (\lambda y.x\{t/\text{🔑}\}\{\text{⅄}i./\text{🔒}\} + y)$$

Implementation: Normalisation

Leads to an interesting normalisation-by-evaluation algorithm.

```
data Env =
  Empty
  | Cons Val Env
  | Lock (Val -> Env)

...

data Neutral
  = ...
  | NVar { level :: Int, keys :: [Val] }
```

Variable lookup means feeding the variable keys to the environment locks.

Prototype at `https://github.com/mvr/tiny`

Final Thoughts

- Easy to tweak the notion of fibration
  - Which definition of equivalence is fastest?
  - Equivariant fibrations ([ACCRS24])?
  - Directed fibrations ([WL20])?
  - Or several notions at once?
- Hand-crafted fibrancy structures? (Thinking of $\mathbb{Z} = \mathbb{Z}$)
- Lazy normalisation algorithm allows more sharing?
- Other non-cubical applications:
  - Myers on form classifiers and connections
  - Fiore et al. on variable binding in HOAS

Thanks!

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