Homotopy Type Theory Electronic Seminar Talks
the 4^{th} of December 2025

Different descriptions of the semantics of computation axioms

parts of the talk are based on joint work with Daniël Otten
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(University of Nantes)

Content, *roughly*

How to *express* this:

$$\frac{\vdash A : \textsc{Type}}{\begin{array}{c} x, x' : A \vdash x = x' : \textsc{Type} \\ x : A \vdash r(x) : x = x \end{array}} \qquad \frac{\begin{array}{c} \vdash A : \textsc{Type} \\ x, x' : A;\ p : x = x' \vdash C(x, x', p) : \textsc{Type} \\ x : A \vdash q(x) : C(x, x, r(x)) \end{array}}{\begin{array}{c} x, x' : A;\ p : x = x' \vdash \mathsf{J}(q, x, x', p) : C(x, x', p) \\ x : A \vdash \mathsf{J}(q, x, x, r(x)) \equiv q(x) \end{array}}$$

in categorical structures.

The semantics of a dependent type theory can be seen as the class of copies of that theory, i.e. the categorical structures that can *express*, in this sense, the theory. This encoding is typically done using the arrows of these categorical structures.

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- (Some, possibly all) arrows of codomain Γ are types A in context Γ and are usually denoted as $\Gamma.A \to \Gamma$.
- Sections of $\Gamma.A \to \Gamma$ are terms of A in context Γ.

Seely, *Locally cartesian closed categories and type theory*, 1983.

Substitution

If we are given $\Delta \xrightarrow{f} \Gamma$ and $\Gamma.A \to \Gamma$ then the judgement:

$$\Delta \vdash A[f] : \textsc{Type}$$

is represented by the pullback:

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is represented by the unique section of $\Delta.A[f] \to \Delta$ such that:

$$\begin{array}{ccc} \Delta & \xrightarrow{\;f\;} & \Gamma \\ \downarrow & & \downarrow{\scriptstyle t} \\ \Delta.A[f] & \longrightarrow & \Gamma.A \\ \downarrow & \lrcorner & \downarrow \\ \Delta & \xrightarrow{\;f\;} & \Gamma \end{array}$$

commutes.

Approaches to identify a model

In a categorical structure (with enough stuff) one can use this language to formulate type constructors of a theory and hence ask if this structure is or is not a model of a given inference rule.

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- the **syntactic** approach, *encoding type constructors into a model in alignment with the syntax*
- the **(higher) categorical** approach, *characterising type constructors via (higher) universal properties*
- the **homotopy theoretic** approach, that *relies on a primitive notion of weak equivalence to phrase the type formers*

Intensional type constructors (with computation rules)

Intensional identity types

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Dependent sum types

$$\frac{\begin{array}{c} \vdash A : \text{Type} \\ x : A \vdash B(x) : \text{Type} \end{array}}{\begin{array}{c} \vdash \Sigma_{x:A} B(x) : \text{Type} \\ x : A, y : B(x) \vdash \langle x, y \rangle : \Sigma_{x:A} B(x) \end{array}}$$

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Axiomatic type constructors[1] (with computation *axioms*)

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[1] Also known as *weak*, *objective*, *propositional* theory.

How semantics works

We said that in an appropriate structure, e.g. a **display map category**, there are some arrows (display maps, that we can denoted as $\Gamma.A \to \Gamma$) that interpret type judgements $\Gamma \vdash A : \textsc{Type}$; and that term judgements $\Gamma \vdash t : A$ are interpreted as sections $\Gamma \to \Gamma.A$ of the corresponding display map.

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- In the **category theoretic approach** one looks for a 1-dimensional categorical property to give to display maps that *characterises* the type constructor, allowing *a choice function as in the syntactic approach to be induced* by this property. The **homotopy theoretic approach** is similar, but the emphasis is on the family of weak equivalences.

An example: identity types

- **Syntactic approach**.
  For every display map $P_A : \Gamma.A \to \Gamma$, there is a choice of:
  - (*Form Rule*) a display map $\Gamma.A.A^{\blacktriangledown}.\,\mathrm{id}_A \to \Gamma.A.A^{\blacktriangledown}$;

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  If the identity types are extensional. For every display map $P_A : \Gamma.A \to \Gamma$, the arrow $v_A : \Gamma.A \to \Gamma.A.A^{\blacktriangledown}$ (obtained by factoring the pair $(1_{\Gamma.A}, 1_{\Gamma.A})$ through $\Gamma.A.A^{\blacktriangledown}$) is isomorphic to a display map.

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A line of research: *how to adapt this approach to axiomatic type constructors, and hence identify a higher dimensional structure with natural categorical conditions that allow to interpret axiomatic theory.*

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However $\Omega \vdash A[f][g] \equiv A[f[g]]$ and $\Omega \vdash t[f][g] \equiv t[f[g]]$ are derivable: in this sense we do not necessarily have a genuine model.

Hofmann's coherence result

However, in:

Hofmann, *On the Interpretation of Type Theory in Locally Cartesian Closed Categories*, 1994.

every finitely complete category is shown to be equivalent to a *split* display map category (still endowed with extensional $=$, and Σ), where 'split' means that there is a choice of pullback squares and $A[f][g] \equiv A[f[g]]$.

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In general, we have a **right-adjoint splitting theorem**: the inclusion:

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- (under some pseudo-stability condition) preserves extensional $=$ and Σ structure.

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Warren, *Homotopy Theoretic Aspects of Constructive Type Theory*, 2008.

Streicher, *Fibred categories à la Jean Bénabou*, 2018.

Hofmann's coherence result

Clairambault & Dybjer, and Maietti proved that there exists a biequivalence between:

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- (under some weak-stability condition) preserves the semantic intensional $=$ and Σ structure.

Lumsdaine, Warren, *The local universes model*, 2015.

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Some reasons to study axiomatic type theory: broader concept of semantics, conservativity.

Path categories i.e. non-genuine models of axiomatic identity types

A path category $\mathcal{C}$ is a category with a terminal object, a class of fibrations and a class of weak equivalences such that the following properties are satisfied:

1. The composition of two fibrations is a fibration as well.
2. Every pullback of a fibration exists and is a fibration as well.
3. Every pullback of an acyclic fibration is a trivial fibration as well.
4. Weak equivalences satisfy 2-out-of-six.
5. Every isomorphism is a trivial fibration and every trivial fibration has a section.
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7. Every arrow of target a terminal object is a fibration.

Path categories i.e. non-genuine models of axiomatic identity types

A path category $\mathcal{C}$ is a category with a terminal object, a class of fibrations and a class of weak equivalences such that the following properties are satisfied:

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2. Every pullback of a fibration exists and is a fibration as well.
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This statement extends to a result à la Clairambault & Dybjer.

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- the 2-category of the *contextual models of extensional =-types* (+ other constructors)
- the 2-category of finitely complete categories

studied by Seely, Hofmann, Clairambault & Dybjer, and Maietti.

The precise statement

Semantics of $=_{\mathrm{ext}}$

$$\{\text{finitely complete categories and structure preserving functors}\}$$

$$\simeq$$

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Models in input need to be *cloven*. Additionally, they need to have *function variable contexts* (LF condition).

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In every path category, a higher-dimensional category is hidden.

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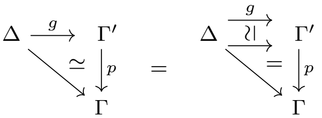
In this setting:

Proposition. Path categories are enriched in groupoids [den Besten] and:

- fibrations $p : \Gamma' \to \Gamma$ are cloven isofibrations:



- path objects are homotopy arrow objects: $(\mathcal{C}/\Gamma)(\Delta, \Gamma')^{\to} \simeq (\mathcal{C}/\Gamma)(\Delta, P_\Gamma \Gamma')$;
- pullbacks are also 2-pullbacks.

In every path category, a higher-dimensional category is hidden.

If we are given a type judgement $\Gamma \vdash A : \textsc{Type}$ and substitutions $\Delta \vdash f : \Gamma$ and $\Delta \vdash g : \Gamma.A$ (i.e. g is given by $g_1 : \Gamma \, , \, g_2 : A[g_1]$) with a context identity proof $\Delta \vdash p : f = g_1$, then the lifted 1-cell is the substitution:

$$\Delta \vdash f : \Gamma \, , \, p^* g_2 : A[f]$$

and the lifted homotopy is the context identity proof $\Delta \vdash f \, , \, p^* g_2 \quad = \quad g_1 \, , \, g_2$ provided by the list:

$$\begin{aligned} \Delta \vdash \quad & p : & f = g_1 \\ \Delta \vdash & \mathsf{r}(p^* g_2) : & p^* g_2 = p^* g_2 \end{aligned}$$

of identity proofs. Now, if $f \equiv g_1$ and $p \equiv \mathsf{r}(g_1)$, then the lifted 1-cell is:

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hence in general the former will not be g and the latter will not be the identity 2-cell: with *axiomatic* identity types we can infer that $\mathsf{r}(g_1)^* g_2 = g_2$ (it is a fragment of the computation axiom for $=$-types) but not that $\mathsf{r}(g_1)^* g_2 \equiv g_2$.

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This is precisely why the display map associated to the type A is a cloven isofibration but not necessarily a normal isofibration.

2-dimensional semantics of axiomatic theories

Display map 2-categories. (2,1)-dimensional categories with a specified class of 1-morphisms, called **display maps**, that satisfy the following conditions:

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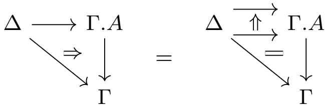

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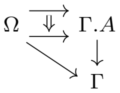

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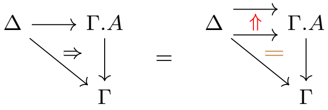
2. To strictify eliminations in 3-4 in change of producing computation *axioms*.



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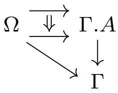

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2-dimensional semantics of axiomatic theories

Theorem. *Display map 2-categories are models of axiomatic dependent type theory.*

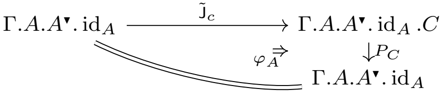
In detail

Under the hypotheses of the elimination rule of identity types, we are able to build a pseudo-term:

$$\begin{array}{ccc} \Gamma.A.A^{\blacktriangledown}.\,\mathrm{id}_A & \xrightarrow{\tilde{\jmath}_c} & \Gamma.A.A^{\blacktriangledown}.\,\mathrm{id}_A\,.C \\ & \varphi_A^{\Rightarrow} & \downarrow p_C \\ & \overset{}{=\!=\!=\!=\!=} & \Gamma.A.A^{\blacktriangledown}.\,\mathrm{id}_A \end{array}$$

In detail

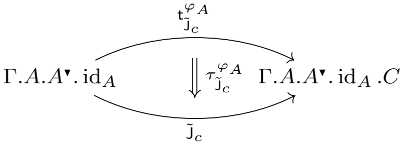
Under the hypotheses of the elimination rule of identity types, we are able to build a pseudo-term:



and, using the cloven isofibration structure on P_C, we obtain a section:

$$\mathsf{t}_{\tilde{\mathsf{J}}_c}^{\varphi_A} : \Gamma.A.A^{\blacktriangledown}.\,\mathrm{id}_A \to \Gamma.A.A^{\blacktriangledown}.\,\mathrm{id}_A\,.C$$

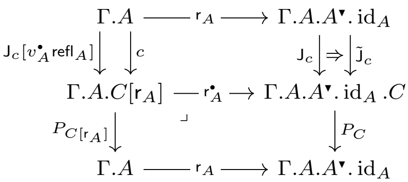
of P_C, at the cost of introducing an additional 2-cell:



We define $\mathsf{J}_c := \mathsf{t}_{\tilde{\mathsf{J}}_c}^{\varphi_A}$.

In detail

Now, referring to the diagram:



we obtain a 2-cell $\mathsf{J}_c[v^{\bullet}_A \mathsf{refl}_A] \Rightarrow c$.

In detail

Now, referring to the diagram:

$$\begin{array}{ccc}
\Gamma.A & \xrightarrow{\quad \mathsf{r}_A \quad} & \Gamma.A.A^{\blacktriangledown}.\,\mathrm{id}_A \\
{\scriptstyle \mathsf{J}_c[v^{\bullet}_A \mathsf{refl}_A]}\big\downarrow \;\; \big\downarrow{\scriptstyle c} & & {\scriptstyle \mathsf{J}_c}\big\downarrow \Rightarrow \big\downarrow{\scriptstyle \tilde{\mathsf{J}}_c} \\
\Gamma.A.C[\mathsf{r}_A] & \xrightarrow{\quad \mathsf{r}^{\bullet}_A \quad} & \Gamma.A.A^{\blacktriangledown}.\,\mathrm{id}_A\,.C \\
{\scriptstyle P_{C[\mathsf{r}_A]}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle P_C} \\
\Gamma.A & \xrightarrow{\quad \mathsf{r}_A \quad} & \Gamma.A.A^{\blacktriangledown}.\,\mathrm{id}_A
\end{array}$$

we obtain a 2-cell $\mathsf{J}_c[v^{\bullet}_A \mathsf{refl}_A] \Rightarrow c$.

Remark. If P_C is normal, then:

$$\mathsf{J}_c\mathsf{r}_A = t^{\varphi_A * \mathsf{r}_A}_{\tilde{\mathsf{J}}_c\mathsf{r}_A} = t^{1_{\mathsf{r}_A}}_{\tilde{\mathsf{J}}_c\mathsf{r}_A} = \tilde{\mathsf{J}}_c\mathsf{r}_A$$

$$(\mathsf{J}_c \Rightarrow \tilde{\mathsf{J}}_c) * \mathsf{r}_A = \tau^{\varphi_A * \mathsf{r}_A}_{\tilde{\mathsf{J}}_c\mathsf{r}_A} = \tau^{1_{\mathsf{r}_A}}_{\tilde{\mathsf{J}}_c\mathsf{r}_A} = 1_{\tilde{\mathsf{J}}_c\mathsf{r}_A}$$

implying that $\mathsf{J}_c[v^{\bullet}_A \mathsf{refl}_A]$ is in fact c.

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we obtain a 2-cell $\mathsf{J}_c[v^{\bullet}_A \mathsf{refl}_A] \Rightarrow c$.

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implying that $\mathsf{J}_c[v^{\bullet}_A \mathsf{refl}_A]$ is in fact c.

However, if P_C is just cloven, then $\mathsf{J}_c[v^{\bullet}_A \mathsf{refl}_A]$ and c can remain different.

An application:

A revisitation of the groupoid model.

We consider the (2,1)-category GRPD of groupoids, functors, and natural transformations (i.e. natural isomorphisms) with **Grothendieck constructions of *pseudofunctors*** $\mathbf{\Gamma \to GRPD}$ as display maps over Γ.

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In particular, the judgemental computation rule for intensional identity type constructor is independent of the axiomatic dependent type theory.

Thank you!