Constructive and Predicative Locale Theory in Univalent Foundations

HoTTEST

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This talk

- A **constructive** and **predicative** framework for locale theory in HoTT/UF.
- Use of this framework for the development of constructive and predicative versions of the following in HoTT/UF:
    1. Stone duality for spectral locales,
    2. Patch locale of a spectral locale, which coreflects Stone locales into spectral ones,
    3. Point-free topology of Scott domains.

A **locale** is a notion of space defined solely by its lattice of opens.

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- Example: the **Tychonoff Theorem**.
- Its point-set manifestation is **equivalent** to the Axiom of Choice [Kel50].
- Its point-free manifestation is **constructively provable** [Coq03].

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We want to **test** HoTT/UF as a foundational setting for constructive mathematics.

Foundations

Notions of size

Definition ($\mathcal{V}$-smallness)

A type $X : \mathcal{U}$ is called **$\mathcal{V}$-small** if it has a specified copy in universe $\mathcal{V}$, i.e.

$$\Sigma_{(Y : \mathcal{V})} X \simeq Y.$$

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Definition (Local $\mathcal{V}$-smallness)

A type $X : \mathcal{U}$ is called **locally $\mathcal{V}$-small** if the identity type $x = y$ is $\mathcal{V}$-small for every pair of inhabitants $x, y : X$.

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Predicative mathematics is a branch of constructive mathematics.

- LEM implies that $\Omega_{\mathcal{U}} \simeq 2_{\mathcal{U}}$, which can be resized into every universe.

Univalence and the propositionality of being small

The propositionality of **being a small type** (for all universes) is equivalent to univalence.

Propositions 2.8 and 2.9 of [dJE23]

The following are equivalent:

- For every type $A : \mathcal{U}$, the type expressing that A is $\mathcal{V}$-small is a proposition (for every pair of universes $\mathcal{U}$ and $\mathcal{V}$).
- The univalence axiom holds.

Some terminological remarks

Throughout this talk, we use the following terminology:

- *There is some* means unspecified existence.
- *There is a chosen/specified* means specified existence.
- We avoid the term *mere existence*.
- For extra clarity, we also say *there is an unspecified* sometimes.

Set replacement

Definition (Set replacement axiom) [Rij22, Axiom 18.1.8]

The **set replacement principle** states that, for every function $f : X \to Y$, if we have that

- Y is a set,
- X is $\mathcal{U}$-small, and
- Y is locally $\mathcal{V}$-small,

then the type image(f) is $(\mathcal{U} \sqcup \mathcal{V})$-small.

We will be interested in the following special case of this axiom:

Proposition (assuming set replacement)

Let $X : \mathcal{U}$ be a type and $Y : \mathcal{U}^+$, a set. For every function $f : X \to Y$, to show that image(f) is $\mathcal{U}$-small it suffices to show that Y is locally $\mathcal{U}$-small.

Theorem 2.11.24 of de Jong [dJon23]

Set replacement is logically equivalent to the existence of small set quotients.

Basic point-free topology

Directed families

Definition (Family)

The type of $\mathcal{W}$**-families over a type** $A : \mathcal{U}$ is

$$\mathsf{Fam}_{\mathcal{W}}(A) :\equiv \Sigma_{(I : \mathcal{W})} I \to A.$$

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Definition (Directed family)

A family $\alpha : I \to X$, where X is a preorder, is called **directed** if

- I is inhabited, and
- for every pair of indices $i, j : I$, there is some $k : I$ such that $\alpha(i) \leq \alpha(k)$ and $\alpha(j) \leq \alpha(k)$.

Finiteness

Definition (Kuratowski finiteness)

A type X is **Kuratowski finite** if there is some surjection $e : \mathrm{Fin}(n) \twoheadrightarrow X$, for some natural number $n : \mathbb{N}$.

Definition (Kuratowski-finite subset)

A subset $S : X \to \Omega_{\mathcal{U}}$ is called **Kuratowski finite** if the subtype $\Sigma_{(x:X)} S(x)$ is a Kuratowski-finite type.

Definition (Frame)

A $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-**frame** consists of

- a type $A : \mathcal{U}$,
- a partial order $- \leq - : A \to A \to \Omega_{\mathcal{V}}$,
- a top element $\mathbf{1} : A$,
- a binary meet operation $- \wedge - : A \to A \to A$,
- a join operation $\bigvee \_ : \mathrm{Fam}_{\mathcal{W}}(A) \to A$;
- satisfying distributivity i.e. $x \wedge \bigvee_{i : I} y_i = \bigvee_{i : I} x \wedge y_i$ for every $x : A$ and $\mathcal{W}$-family $(y_i)_{i:I}$ in A.

Sethood follows from the partial order.

Large, **locally small**, and **small-complete frame**: $(\mathcal{U}^+, \mathcal{U}, \mathcal{U})$-frame.

Frames in HoTT/UF

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Large, **locally small**, and **small-complete frame**: $(\mathcal{U}^+, \mathcal{U}, \mathcal{U})$-frame.

We do not need the universe generality involved in the above definition but we define it for now so that we can explain *why* we do not need it.

The powerset frame

Definition (Powerset frame)

For every set $A : \mathcal{U}$, the **powerset frame** on A is given by:

- Carrier: $A \to \Omega_{\mathcal{U}}$.
- Inclusion order: $\Pi(a : A)S(a) \to T(a)$.
- Finite meets and arbitrary joins: subset intersection and union.

This is large, locally small, and small complete.

No-go theorem for complete, small lattices

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Theorem

If there exists a nontrivial small frame then Ω-resizing holds.

The right category of frames

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- We use the variable letters $X, Y, \ldots$ for locales, and $U, V, \ldots$ for their elements.
- For continuous maps of locales, we use $f : X \to Y$.
- The defining frame homomorphism of a continuous map $f : X \to Y$ is denoted $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$.

On local smallness

We used the term “locally small” in two different contexts, which we have not yet justified.

Proposition

For every $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame L, and for every universe $\mathcal{T}$, the carrier set of L is locally $\mathcal{T}$-small if and only if the proposition $x \leq y$ is $\mathcal{T}$-small.

Corollary

For every $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame L, the proposition $x \leq y$ is at most $\mathcal{U}$-small.

So the largest category one has to consider is $(\mathcal{U}^+, \mathcal{U}^+, \mathcal{U})$-frames

Question: can it not be the case that local smallness follows from other properties?

On local smallness (continued)

Let $X : \mathcal{U}^+$ be a large set.

Observe that the powerset frame $\mathcal{P}(X) \equiv X \to \Omega_{\mathcal{U}}$ is **large** and **small complete**.

Proposition

If the powerset frame over a large set $X : \mathcal{U}^+$ is locally small, then $\Omega_{\neg\neg}$-resizing holds

Proof

- Take a large proposition $P : \mathcal{U}^+$.
- Consider the powerset frame $\mathcal{P}(P)$.
- By local smallness, $\top_P \subseteq \emptyset$ is small, and it is equivalent to $\neg P$.
- This means $\neg P$ is $\mathcal{U}$-small.
- It follows that $\neg\neg P$ is $\mathcal{U}$-small.

So this is an example of a frame that is (1) **large**, and (2) **small complete**, but not **locally small** unless $\Omega_{\neg\neg}$-resizing holds.

The terminal locale (in $\mathbf{Loc}_{\mathcal{U}}$)

Definition (Terminal locale)

The **terminal** locale, denoted $\mathbf{1}_{\mathcal{U}}$, is given by the following frame:

- Carrier set is $\Omega_{\mathcal{U}}$.
- Propositions ordered under implication: $P \leq Q :\equiv P \Rightarrow Q$.
- Finite meets are products of propositions.
- Join-completeness is exactly the *unspecified existence* operator.

Proposition

This is the initial object in $\mathbf{Frm}_{\mathcal{U}}$ i.e. the terminal object in $\mathbf{Loc}_{\mathcal{U}}$.

The Sierpiński locale (in $\mathrm{Loc}_{\mathcal{U}}$)

Definition (Universal property of Sierpiński)

A locale X has the **universal property of Sierpiński** if there is a specified open $U : \mathcal{O}(X)$ satisfying the property that, for every locale Y and every open $V : \mathcal{O}(Y)$, there is a unique map $f : Y \to X$ satisfying $V = f^*(U)$.

The above property says that the locale in consideration is *freely generated* by the unit type.

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Definition (The Sierpiński locale)

The **Sierpiński locale** $\mathbb{S}_{\mathcal{U}}$ is given by the following frame:

- Carrier: set of **Scott-continuous** maps $S : \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}}$.
- Order: inclusion.
- Meets and joins: intersection and union.

We define the map $\mathsf{true} : \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}}$ as $\mathsf{true}(P) :\equiv P$ holds.

The Sierpiński locale (continued)

- The Sierpiński frame is a subframe of the powerset frame over a large set: $\Omega_{\mathcal{U}}$.
- This means that the inclusion order $S \leq T :\equiv \Pi(P : \Omega_{\mathcal{U}})\, S(P) \to T(P)$ is *a priori* large.
- But it can be shown to be small-valued as it is equivalent to

$$S \leq_0 T :\equiv \Pi(b : 2)\, S(\beta(b)) \to T(\beta(b)),$$

  where $\beta : 2 \to \Omega_{\mathcal{U}}$ sends 0 to $\bot$ and 1 to $\top$.
- Every proposition P can be expressed as the directed join:

$$\bigvee \{\beta(b) \mid b : 2, \beta(b) \leq P\}.$$

- By Scott continuity:

$$S\left(\bigvee \{\beta(b) \mid b : 2, \beta(b) \leq P\}\right) = \bigvee \{S(\beta(b)) \mid b : 2, \beta(b) \leq P\}.$$

Sublocales and nuclei

In traditional locale theory, there are several equivalent ways to define the notion of sublocale:

1. Regular frame quotients (isomorphism classes of surjections),
2. Frame congruences,
3. Nuclei,
4. Fixsets, which are subsets that are (1) closed under arbitrary meets and (2) form exponential ideals with respect to Heyting implication.

Nuclei axiomatize monads arising from the composition of frame quotients $q : \mathcal{O}(X) \twoheadrightarrow \mathcal{O}(Y)$ with their right adjoints $q_* : \mathcal{O}(Y) \to \mathcal{O}(X)$.

Definition (Nucleus)

A **nucleus** on locale X is a function $j : \mathcal{O}(X) \to \mathcal{O}(X)$ that is

- inflationary,
- idempotent, and
- preserves binary meets.

Impredicativity in locale theory

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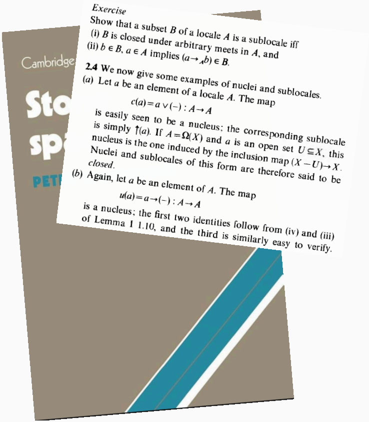

Exercise
Show that a subset B of a locale A is a sublocale iff
(i) B is closed under arbitrary meets in A, and
(ii) $b \in B$, $a \in A$ implies $(a \to_A b) \in B$.

2.4 We now give some examples of nuclei and sublocales.
(*a*) Let a be an element of a locale A. The map

$$c(a) = a \vee (-) : A \to A$$

is easily seen to be a nucleus; the corresponding sublocale is simply $\uparrow(a)$. If $A = \Omega(X)$ and a is an open set $U \subseteq X$, this nucleus is the one induced by the inclusion map $(X - U) \to X$. Nuclei and sublocales of this form are therefore said to be *closed*.
(*b*) Again, let a be an element of A. The map

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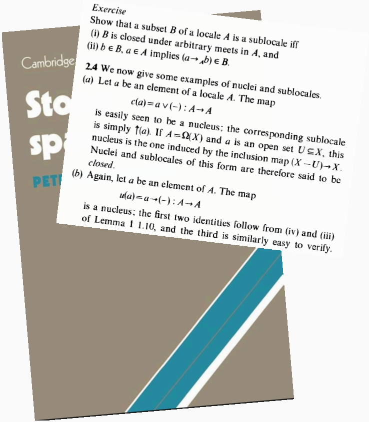

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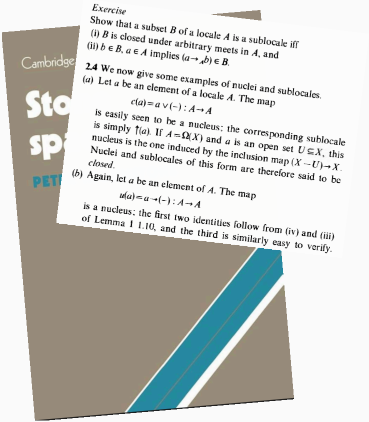

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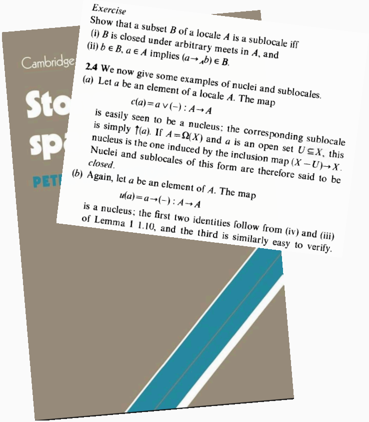
3. More generally, **right adjoints** of frame homomorphisms.
4. **Arbitrary meets**.
5. **Fixsets**, the definition of which presupposes the existence of arbitrary meets and Heyting implications.



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(*b*) Again, let a be an element of A. The map

$$u(a) = a \to (-) : A \to A$$

is a nucleus; the first two identities follow from (iv) and (iii) of Lemma I 1.10, and the third is similarly easy to verify.

Notion of bases for locales

Definitions (Intensional/Extensional base)

A family $(B_i)_{i:I}$ of opens forms a **base** for locale X if

for every $U : \mathcal{O}(X)$, there is a specified, directed, small family $(i_j)_{j:J}$ on the base index satisfying $U = \bigvee_{j:J} B_{i_j}$.

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Called **extensional** if the family in consideration is an embedding.

We use the term **basic covering family**.

Some classes of locales

Definition (Spectral locale [†])

A locale X is **spectral** if it has some small base $(B_i)_{i:I}$ satisfying the following three conditions:

- It consists of **compact opens**.
- It contains the **top open**.
- For every pair of indices $i, j : I$, there is some $k : I$ such that $B_k = B_i \wedge B_j$.

Definition (Zero-dimensional locale)

A locale X is called **zero-dimensional** if its frame $\mathcal{O}(X)$ has some small base

- that **consists of clopens**.

Definition (Regular locale)

A locale X is called **regular** if its frame $\mathcal{O}(X)$ has some small base such that

- the basic covering families **consist of elements well inside their joins**.

Definition (Locally compact locale)

A locale X is called **locally compact** if its frame $\mathcal{O}(X)$ has some small base such that

- the basic covering families **consist of elements way below their joins**.

Posetal adjoint functor theorem for frames

Lemma

Let K and L be two frames, and let $h : K \to L$ be a frame homomorphism. If K has an unspecified small, weak base, then h has a **right adjoint**.

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Every frame with an unspecified, small, weak base has Heyting implications.

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Given a continuous map $f : X \to Y$, we denote

- by $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ its defining frame homomorphism (as previously mentioned),
- by $f_* : \mathcal{O}(X) \to \mathcal{O}(Y)$ the right adjoint of f^* (provided that Y has some small, weak base).

Examples of small bases

Example (Terminal locale)

The family $\beta : \mathbf{2} \to \Omega_{\mathcal{U}}$, sending 0 to $\bot$ and 1 to $\top$, forms a base for the terminal locale $\mathbf{1}_{\mathcal{U}}$.

- Each proposition $P : \Omega_{\mathcal{U}}$ can be expressed as:

$$\bigvee \{\beta(b) \mid b : 2, \beta(b) \leq P\}.$$

Example (Sierpiński locale)

The family $(\uparrow\beta(b))_{b:\mathbf{2}}$ forms a base for the Sierpiński locale. Every Scott-open subset $S : \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}}$ is expressible as

$$S = \bigvee_{b\,:\,\mathbf{2}} \{\uparrow\beta(b) \mid \beta(b) \in S\}.$$

Brief comparison with the formal-topological approach

- We work with large, locally small, and small-complete locales.
- In formal topology, one replaces the notion of **frame** with **formal topology** as a notion of space in itself.
- These are small sets of generators equipped with a covering relation.
- The frames they generate are always large, locally small, and small complete.
- Working with formal topologies is analogous to the **specified** existence of **intensional** bases.

Compact and spectral locales

Compact locales

Theorem

For every open $U : \mathcal{O}(X)$, the following conditions are equivalent:

- For every family of opens $(V_i)_{i:I}$ with $U \leq \bigvee_{i:I} V_i$, there is a Kuratowski finite subfamily $(V_{i_j})_{j:J}$ such that $U \leq \bigvee_{j:J} V_{i_j}$.
- For every family of opens $(V_i)_{i:I}$ with $U = \bigvee_{i:I} V_i$, there is a Kuratowski finite subfamily $(V_{i_j})_{j:J}$ such that $U = \bigvee_{j:J} V_{i_j}$.
- For every directed family $(V_i)_{i:I}$ with $U \leq \bigvee_{i:I} V_i$, there is some index $i : I$ such that $U \leq V_i$.

Definition (Compact open)

An open $U : \mathcal{O}(X)$ is called **compact** if satisfies the equivalent conditions of the above theorem.

Definition (Compact locale)

A locale X is called **compact** if its top open 1_X is compact.

Examples of compact locales

- The terminal locale is compact.
- The Sierpiński locale is compact.
- The discrete locale on $\mathcal{P}(\mathbb{N})$ is not compact.

Stone spaces

Stone spaces $\simeq$ Boolean algebras

Motivation for spectral locales

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Motivation for spectral locales

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Spectral spaces	$\simeq$	Distributive lattices

- Stone [Sto36] first discovered this duality in the context of Boolean algebras.
- He then generalized [Sto37] it to distributive lattices.

Motivation for spectral locales (continued)

Let X be a **Stone space**.

Let L be a **Boolean algebra**.

Motivation for spectral locales (continued)

Let X be a **Stone space**.

The clopens of X form a **Boolean algebra**.

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It is really the *algebra of compact opens* that matters!

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Let X be a **spectral space**.

Let L be a **distributive lattice**.

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Let X be a **spectral space**.

The compact opens of X form a **distributive lattice**.

Let L be a **distributive lattice**.

The prime filters of L form a **spectral space**.

Spectral locales in HoTT/UF

Definition (Spectral locale [‡])

A locale X is called **spectral** if it satisfies the following conditions:

- **(SP1)** It is compact (i.e. the empty meet is compact).
- **(SP2)** Compact opens are closed under binary meets.
- **(SP3)** The family $\mathsf{K}(X) \hookrightarrow \mathcal{O}(X)$ forms a **weak** base.
- **(SP4)** The type $\mathsf{K}(X)$ is **small**.

Definition (Spectral map)

A continuous map $f : X \to Y$ is called **spectral** if it reflects compact opens i.e. $f^*(K)$ is a compact open of X, for every compact open $K : \mathcal{O}(Y)$.

Spec$_{\mathcal{U}}$: the category of spectral locales and spectral maps.

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Lemma

Univalence implies that being spectral (as in ‡) is a proposition.

Univalence *seems to be required to express the* ***property*** *of being spectral!*

On the two definitions of spectrality

- I already gave a definition of spectrality (in †) that is a proposition.
- But the main desideratum for a candidate notion of spectrality is that it enjoys **Stone duality**.
- Definition † is a **property** without using univalence **but** it does **not** enjoy Stone duality.
- Definition ‡ enjoys Stone duality **but** its definition *seems to* require univalence.
- The two are equivalent, but for the proof, we use the propositionality of Definition ‡.

Characterization of spectral locales

Definition (Spectral base)

We denote by $\mathsf{SpectralBaseStr}(X)$ the type of small bases satisfying the conditions:

- consists of compact opens,
- contains the top open,
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Definition † is the truncation of $\mathsf{SpectralBaseStr}(X)$.

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Lemma

For every X with a specified small base $(B_i)_{i:I}$, every compact open of X is in the base.

Corollary

For every locale X with a specified small base $(B_i)_{i:I}$, if the base consists of compact opens, then we have an equivalence: $\mathrm{image}(B) \simeq \mathsf{K}(X)$.

Corollary

Assuming **set replacement**, having a specified small base implies that the type $\mathsf{K}(X)$ is small.

Characterization of spectral locales (continued)

Assuming **univalence** and **set replacement**:

Lemma

For every unspecified intensional, spectral, small base on a locale X, we can obtain a specified extensional, spectral, small base.

In the terminology of [KECA17], the type SpectralBaseStr(X) has **split support**:

$$\|\text{SpectralBaseStr}(X)\| \to \text{SpectralBaseStr}(X).$$

Theorem

Definition † and Definition ‡ are equivalent.

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Note: the propositionality of **being spectral** is crucial here.

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From Tom de Jong's PhD thesis:

> **Lemma 4.8.3.** *If a dcpo comes equipped with a small compact basis, then it is structurally algebraic. Hence, if a dpco has an unspecified small compact basis, then it is algebraic.*

Stone duality

Definition (Small distributive lattice)

A **distributive $\mathcal{U}$-lattice** consists of

- a set $|L| : \mathcal{U}$,
- elements $\mathbf{0}, \mathbf{1} : |L|$,
- operations $- \wedge - : |L| \to |L| \to |L|$ and $- \vee - : |L| \to |L| \to |L|$,
- satisfying the laws of **associativity**, **commutativity**, **unitality**, **idempotence**, and **absorption**.

$\mathbf{DLat}_{\mathcal{U}}$: the category of small distributive lattices over base universe $\mathcal{U}$.

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Recent result due to David Wärn: sethood condition in the above definition is not essential!

Definition (Ideal)

A $\mathcal{U}$**-ideal** of a distributive lattice L is a subset $I : L \to \Omega_{\mathcal{U}}$ that is

- inhabited,
- downward closed,
- closed under binary joins.

Stone duality for spectral locales

$$\mathbf{Spec}_{\mathcal{U}} \underset{\mathsf{Spec}}{\overset{\mathsf{K}}{\rightleftarrows}} \mathbf{DLat}_{\mathcal{U}}$$

Lemma

For every small distributive lattice L, we have a spectral locale $\mathsf{Spec}(L)$, defined by the frame $\mathsf{Idl}(L)$.

Lemma

For every spectral locale X (as in ‡), the type $\mathsf{K}(X)$ is a small **distributive lattice**.

Theorem

The maps K and Spec form an equivalence of categories so we have an equivalence:

$$\mathbf{Spec}_{\mathcal{U}} \simeq \mathbf{DLat}_{\mathcal{U}}.$$

This is categorical equivalence. Using univalence and the SIP on the categories, type equivalence also follows.

Patch locale of a spectral locale

Stone locales

Definition (Clopen)

A **clopen** of a locale X is an open $U : \mathcal{O}(X)$ that has a Boolean complement (in the frame $\mathcal{O}(X)$).

Definition (Zero-dimensional locale)

A locale is called **zero-dimensional** if has some small base consisting of clopens.

Definition (Stone locale)

A **Stone locale** is one that is compact and zero-dimensional.

Lemma

In Stone locales, the clopens and the compact opens coincide.

Relationship between spectral and Stone locales

Lemma

Every Stone locale is spectral.

What can be said about the other direction?

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Every spectral locale can be **universally transformed** into a Stone one using the patch topology.

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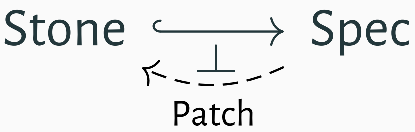

We use Escardó's definition of the patch locale based on the frame of Scott continuous nuclei.

The frame of Scott-continuous nuclei vs. the frame of all nuclei

A nucleus is called **Scott continuous** if it preserves joins of directed families.

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Escardó's previous work [Esc99; Esc01] exploits the fact that the patch frame is a subframe of the frame of **all** nuclei.

In our foundational setting, we **do not know** if the frame of all nuclei can be constructed. Is it locally small? Does it have a small base?

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Conjecture

If the meet-semilattice of **all nuclei** is join-complete, then some form of propositional resizing holds.

The patch locale in HoTT/UF

Because the frame of all nuclei does not seem to be available, many technical lemmas had to be reworked whilst proving the universal property of patch in this foundational setting.

- One of those issues was solved by our collaborator Igor Arrieta.
- For the details of the patch locale, see [AET25].

Topology of algebraic DCPOs

Algebraic DCPOs and Scott domains

Definition (Algebraic DCPO, cf. [dJon23, Definition 4.6.3])

A DCPO D (over base universe $\mathcal{U}$) is called **algebraic** if it satisfies the following conditions:

- The type $\mathsf{K}(D)$ forms a **weak** base.
- The type $\mathsf{K}(D)$ is small.

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Definition (Scott domain)

A **Scott domain** is a pointed algebraic DCPO D satisfying the following two conditions:

- *Small-bounded-completeness*: every small family that has an upper bound, also has a least upper bound.
- *Decidability of upper boundedness for compact elements*: the type $c \uparrow d$ is **decidable**, for every pair of elements $c, d : \mathsf{K}(D)$.

The Scott locale

Definition (Scott open subset)

A subset $S : D \to \Omega_{\mathcal{U}}$ is called **Scott open** if

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Lemma

For every DCPO D, the Scott open subsets form a frame.

- We denote this by $\sigma(D)$ and the locale it defines by $\mathrm{Scott}(D)$.

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We prove the above theorem by proving the spectrality of the following small base:

- $\gamma : \mathsf{List}(\mathsf{K}(D)) \to \sigma(D)$
- $\gamma(c_0, \ldots, c_{n-1}) :\equiv {\uparrow} c_0 \cup \cdots \cup {\uparrow} c_{n-1}$.

Conclusion and Further Work

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  - It is part of TYPETOPOLOGY:

    https://martinescardo.github.io/TypeTopology/Locales.index.html

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  - It was previously done by Coquand and Zhang [CZ03], but using formal topology.

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  - Does not seem to be possible, but no one has shown that it is impossible.
- Open problem: show that the construction of the **frame of all nuclei** relies on some form of impredicativity.
- Investigate the relationship between Patch(Scott(D)) and the Lawson topology.
- Extend this work to **stably locally compact locales** and their patch.
  - It was previously done by Coquand and Zhang [CZ03], but using formal topology.
- Priestley duality is also known to work constructively.

Further work

- Construct more interesting locales using **formal topology**.
- Open problem: construct the product of two locales predicatively.
  - Does not seem to be possible, but no one has shown that it is impossible.
- Open problem: show that the construction of the **frame of all nuclei** relies on some form of impredicativity.
- Investigate the relationship between Patch(Scott(D)) and the Lawson topology.
- Extend this work to **stably locally compact locales** and their patch.
  - It was previously done by Coquand and Zhang [CZ03], but using formal topology.
- Priestley duality is also known to work constructively.
  - Does it work predicatively like Stone duality?

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