# Directed univalence and the Yoneda embedding for synthetic $(\infty, 1)$-categories

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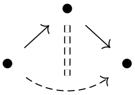
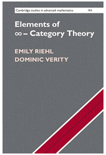
***In memory of Thomas Streicher (1958–2025)***

# The concept of $(\infty, 1)$-category

- $(\infty, 1)$**-categories:** weak composition of $1$-morphisms given uniquely *up to contractibility*



- How to express this in HoTT?
- *Problem:* We have path types $(a =_A b)$, but what about directed hom types $(a \to_A b)$?
- Several possible type-theoretic frameworks, *e.g.* by Warren, Licata–Harper, Annenkov–Capriotti–Kraus–Sattler, Nuyts, North, Weaver–Licata, Altenkirch–Neumann, . . .
- Other synthetic theories: Riehl–Verity, Cisinski–Cnossen–Nguyen–Walde, Martini–Wolf
- **In our work:** Riehl–Shulman's *simplicial type theory* (2017). Also heavily influenced by Riehl–Verity's $\infty$-cosmos theory (2013-2021-. . . ).





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HIGHER STRUCTURES

A type theory for synthetic $\infty$-categories

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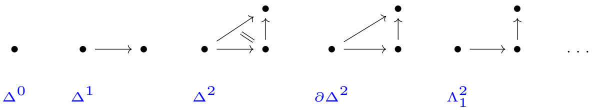
# Simplicial HoTT

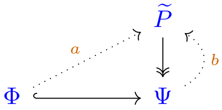
1. **Simplicial HoTT:** Extension of HoTT by Riehl–Shulman '17
2. add strict shapes




3. add extension types (due to Lumsdaine–Shulman, cf. Cubical Type Theory):

**Input:**

- shape inclusion $\Phi \hookrightarrow \Psi$
- family $P : \Psi \to \mathcal{U}$
- partial section $a : \Pi_{t:\Phi} P(t)$

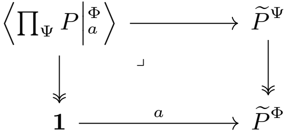



$\rightsquigarrow$

**Extension type** $\left\langle \prod_\Psi P \Big|^\Phi_a \right\rangle$

with terms $b : \Pi_\Psi P$ such that $b|_\Phi \equiv a$.

Semantically:



# Hom types I

**Definition (Hom types, [RS17])**

Let $B$ be a type. Fix terms $a, b : B$. The type of *arrows in* $B$ *from* $a$ *to* $b$ is the extension type

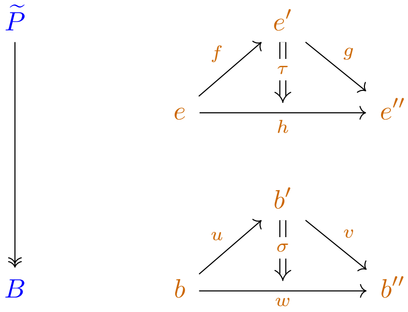
$$\hom_B(a, b) :\equiv (a \to_B b) :\equiv \left\langle \Delta^1 \to B \Big|^{\partial\Delta^1}_{[a,b]} \right\rangle.$$

**Definition (Dependent hom types, [RS17])**

Let $P : B \to \mathcal{U}$ be family. Fix an arrow $u : \hom_B(a, b)$ in $B$ and points $d : P\,a$, $e : P\,b$ in the fibers. The type of *dependent arrows in* $P$ *over* $u$ *from* $d$ *to* $e$ is the extension type

$$\mathrm{dhom}_{P,u}(d, e) :\equiv (d \to^P_u e) :\equiv \left\langle \prod_{t:\Delta^1} P(u(t)) \Big|^{\partial\Delta^1}_{[d,e]} \right\rangle.$$

# Hom types II



# Segal, Rezk, and discrete types

We can now define synthetic $(\infty, 1)$-categories using shapes and extension types:

**Definition (Synthetic $(\infty, 1)$-categories, [RS17])**

- **Synthetic pre-$(\infty, 1)$-category *aka* Segal type:** types $A$ with *weak composition*, *i.e.*:

$$\iota : \Lambda_1^2 \hookrightarrow \Delta^2 \rightsquigarrow A^\iota : A^{\Delta^2} \xrightarrow{\simeq} A^{\Lambda_1^2} \qquad \text{(Joyal).}$$

- **Synthetic $(\infty, 1)$-category *aka* Rezk type:** Segal types $A$ satisfying *Rezk completeness/local univalence*, *i.e.*

$$\mathrm{idtoiso}_A : \Pi_{x,y:A}(x =_A y) \xrightarrow{\simeq} \mathrm{iso}_A(x, y).$$

- **Synthetic $\infty$-groupoid *aka* discrete type:** types $A$ such that *every arrow is invertible*, *i.e.*

$$\mathrm{idtoarr}_A : \Pi_{x,y:A}(x =_A y) \xrightarrow{\simeq} \hom_A(x, y).$$

# Adequate semantics of synthetic $\infty$-category theory

**Theorem (Shulman '19, Riehl–Shulman '17)**

1. *Every $\infty$-topos admits a model of HoTT.*
2. *Every $\infty$-topos of simplicial objects admits a model of sHoTT, with weakly stable extension types.*

**Theorem (W '21)**

*Extension types are strictly substitution-stable.*

**Corollary**

1. *Synthetic $\infty$-category theory interprets to ordinary $\infty$-category theory.*
2. *Synthetic $\infty$-category theory interprets to internal $\infty$-category theory (cf. Martini–Wolf, Cisinski–Ngyuen–Walde–Cnossen, Rasekh, Stenzel).*

# Properties of Segal types

In [RS17] it is shown that:

- The **hom-types** of a Segal type are **groupoidal** (*aka discrete*).
- Discrete types are those types all of whose arrows are invertible (automatically Rezk).
- **Closure properties** from orthogonality characterizations, cf. also [BW23]

# Functors and natural transformations

- Segal types have **categorical structure**: composition $g \circ f$, identities $\mathrm{id}_x$, and homotopies
$$h \circ (g \circ f) = (h \circ g) \circ f, \quad \mathrm{id}_y \circ f = f, \quad f \circ \mathrm{id}_x = f.$$
- Any map $f : A \to B$ between Segal types is automatically a **functor**.
- For $f, g : A \to B$ define the type of **natural transformations** as
$$(f \Rightarrow g) :\equiv \hom_{A \to B}(f, g) :\equiv \left\langle \Delta^1 \to (A \to B)\Big|^{\partial\Delta^1}_{[f,g]} \right\rangle.$$
- One can then *prove* that for $\varphi : (f \Rightarrow g)$ any arrow $u : x \to_A y$ gives rise to the expected naturality square:

$$\begin{array}{ccc} fx & \xrightarrow{\varphi_x} & gx \\ {\scriptstyle fu}\downarrow & & \downarrow{\scriptstyle gu} \\ fy & \xrightarrow[\varphi_y]{} & gy \end{array}$$

# Cocartesian type families

- Any type family $P : B \to \mathcal{U}$ **transforms covariantly** in paths:
  $u : a =_B b \quad \rightsquigarrow \quad u_! : P\,a \to P\,b$
- What about the **directed** analogue? $u : a \to_B b \quad \rightsquigarrow \quad u_! : P\,a \to P\,b$
- **Cocartesian families:** $\infty$-categories parametrized over an $\infty$-category

**Definition (Cocartesian family, Buchholtz–W '21)**

A type family $P : B \to \mathcal{U}$ is *cocartesian* if every arrow in $B$ universally lifts to a $P$-dependent arrow.

**Theorem (Buchholtz–W '21)**

*Lifting and transport in cocartesian families can be expressed via (fibered) adjoint functors à la Street.*

**Theorem (Closure properties of cocartesian families, Buchholtz–W '21)**

*Synthetic cocartesian fibrations form an $\infty$-cosmos in the sense of Riehl–Verity.*

# Covariant type families

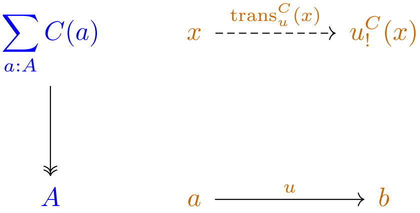
**Definition (Covariant family, [RS17])**

Let $C : A \to \mathcal{U}$ be a family. It is *covariant* if and only if for all $a, b : A$, arrows $u : (a \to_A b)$ and points $x : C(a)$ the type

$$\sum_{y:C(b)} (x \to^C_u y)$$

is contractible.

This give a synthetic analogue of discrete covariant or *left* fibrations:



# Covariant type families: Functoriality & naturality

- Let $C : A \to \mathcal{U}$ be a covariant family and $A$ be Segal.
- If $A$ is Segal, then $\widetilde{C} :\equiv \sum_{a:A} C(a)$ is.
- **Discreteness:** Each fiber $C(x)$ is discrete.
- **Functoriality:** Lifting gives a family of maps $\mathrm{lift}^C : \prod_{x,y:A} (x \to_A y) \to C(x) \to C(y)$ with $\mathrm{lift}^C(f,u) :\equiv f_! u$. For $f : (x \to_A y)$, $g : (y \to_A z)$, and $u : C(x)$ we have identifications

$$g_!(f_! u) = (gf)_! u \quad (\mathrm{id}_x)_! u = u.$$

- **Naturality:** Assume $C, D : A \to \mathcal{U}$ are covariant. Let $\varphi : \prod_{x:A} C(x) \to D(x)$. Then, for any $f : (x \to_A y)$ and $u : C(x)$. Then we have an identification:

$$\varphi_y(f_! u) = f_!(\varphi_x u)$$

- *Example:* For $a : A$, the family $\hom_A(a, -) : A \to \mathcal{U}$ is covariant. For $x : A$, $e : \hom_A(a,x)$ it acts via $f : \hom_A(x,y)$ as $f_! e = f \circ e$.

# Fibered Yoneda lemma as directed path induction

**Theorem (Directed path induction)**

*Fix* $b : B$. *For* $P : \left(\sum_{x:B}(b =_B x)\right) \to \mathcal{U}$ *we have an equivalence:*

$$\left(\prod_{x:B}\prod_{p:b=_B x} P(x,p)\right) \underset{\mathrm{ind}_b^P}{\overset{\mathrm{ev}_{\mathrm{refl}_b}}{\rightleftarrows}} P(b, \mathrm{refl}_b) \quad \simeq$$

**Theorem ((dependent) Yoneda Lemma for covariant families, [RS17])**

*Let* $B$ *be a Segal type, and fix* $b : B$. *For a covariant type family* $P : \left(\sum_{x:B}(b \to_B x)\right) \to \mathcal{U}$, *we have an equivalence:*

$$\left(\prod_{x:B}\prod_{p:b\to_B x} P(x,p)\right) \underset{\mathbf{y}_b^P}{\overset{\mathrm{ev}_{\mathrm{id}_b}}{\rightleftarrows}} P(b, \mathrm{id}_b) \quad \simeq$$

# Fibered Yoneda lemma: proof idea

**Theorem (Yoneda Lemma for covariant families, [RS17])**

*Let $A$ be a Segal type, and $a : A$ any term. For a covariant type family $C : A \to \mathcal{U}$, we have an equivalence:*

$$\mathrm{evid}_a^C : \left( \prod_{x:A} \hom_A(a,x) \to C(x) \right) \xrightarrow{\simeq} C(a)$$

- The inverse map is given by

$$\mathbf{y}_a^C : C(a) \to \left( \prod_{x:A} \hom_A(a,x) \to C(x) \right), \quad \mathbf{y}_a^C(u)(x)(f) :\equiv f_! u$$

- Proof "simply" follows from naturality properties and covariance of $\hom_A(a,-)$.
- There also exists a *dependent version*.
- Both have been formalized in Kudasov's new proof assistant Rzk.
- Cocartesian and other generalizations due to Buchholtz–W and W have been proven, but formalization is WIP.

# The Rzk proof assistant

SRC
- hott
  - 06-contractible.rzk.md
  - 07-fibers.rzk.md
  - 08-families-of-maps.rzk.md
  - 09-propositions.rzk.md
  - 10-trivial-fibrations.rzk.md
- simplicial-hott
  - 03-simplicial-type-theory.rzk.md
  - 04-extension-types.rzk.md
  - 05-segal-types.rzk.md
  - 06-2cat-of-segal-types.rzk.md
  - 07-discrete.rzk.md
  - 08-covariant.rzk.md
  - 09-yoneda.rzk.md
  - 10-rezk-types.rzk.md
  - 12-cocartesian.rzk.md M

OUTLINE

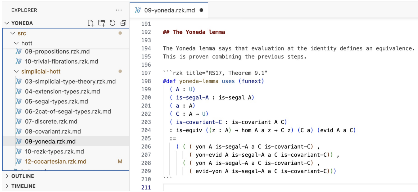
TIMELINE

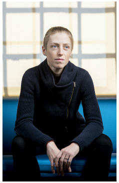
simplicial-hott > 05-segal-types.rzk.md

````

```rzk
#assume funext : FunExt
#assume extext : ExtExt
```


## Hom types

Extension types are used to define the type of arrows
between fixed terms:

```rzk title="RS17, Definition 5.1"
#def hom
  ( A : U)
  ( x y : A)
  : U
  :=
    ( t : Δ¹) →
    A [ t ≡ 0₂ ↦ x ,  -- the left endpoint is exactly x
        t ≡ 1₂ ↦ y]   -- the right endpoint is exactly y
```
````

# Formalizing $\infty$-categories in Rzk

- Kudasov has developed the Rzk proof assistant, implementing sHoTT: `https://rzk-lang.github.io/`
- Using Rzk we initiated the first ever formalizations of $\infty$-category theory.
- In spring 2023, with Kudasov and Riehl we formalized the (discrete fibered) Yoneda lemma of $\infty$-category theory: `https://emilyriehl.github.io/yoneda/`
- alongside many other results
- Many proofs in this $\infty$-dimensional setting *easier* than in dimension 1!
- Formalization helped find a mistake in original paper
- More students & researchers joined us developing a library for $\infty$-category theory: `https://rzk-lang.github.io/sHoTT/`



# Synthetic $\infty$-category theory in sHoTT

- Functors, natural transformations, discrete fibrations & fibered Yoneda lemma, adjunctions (Riehl–Shulman '17)
- Cartesian fibrations (Buchholtz–W '21) & generalizations (W '21)
- Limits and colimits (Bardomiano '22)
- Conduché fibrations (Bardomiano '24)
- Proof assistant `Rzk` (Kudasov '23) and formalization of fibered Yoneda lemma (Kudasov–Riehl–W '23)
- sHoTT library and more formalizations (Abounegm, Bakke, Bardomiano, Campbell, Carlier, Chatzidiamantis-Christoforidis, Ergus, Hutzler, Kudasov, Maillard, Martínez, Pradal, Rasekh, Riehl, F. Verity, Walde, W '23–)

**But many desiderata missing!**

*opposite categories, categories $\mathcal{S}$ and $\mathrm{Cat}$, presheaves & Yoneda embedding, higher algebra,*

*...*

# Multimodal type theory

- **Multi-modal dependent type theory (MTT)** to the rescue! (Gratzer–Kavvos–Nuyts–Birkedal ’20)
- start from a *cubical* outer layer, augmented by an instance of MTT
- the added modal operators: **simplicial localization** $\boxslash$, **opposite** $\mathfrak{o}$, **twisted arrows** $\mathfrak{t}$ **(groupoid) core/discretization** $\flat \dashv$ **codiscretization** $\sharp$, **path type** $(-)^{\Box} \dashv$ **amazing right adjoint** $(-)_{\Box}$
- plus axioms about the interaction between the simplicial and modal structure
- This unlocks a whole new range of constructions
- We call the ensuing type theory *triangulated type theory*

$$\begin{array}{c} \mathcal{S}^{\Delta^{\mathrm{op}}} \\ \downarrow\uparrow\downarrow\uparrow \\ \mathcal{S} \end{array}$$

See also work on cohesive $\infty$-toposes by Schreiber (’13), Shulman (’18), Myers–Riley (’23), as well as internal universes via a tiny interval by Licata–Orton–Pitts-Spitters (’18) and Riley (’24).

# Intuitions for the modalities

- **Opposite** $\mathfrak{o}$: $\langle \mathfrak{o} \mid A \rangle$ has its $n$-simplices reversed
- **Discretization/core** $\flat$: $\langle \flat \mid A \rangle \to A$ is the maximal subgroupoid of $A$
- **Codiscretization** $\sharp$: $A \to \langle \sharp \mid A \rangle$ is localization at $\partial\Delta^n \to \Delta^n$ (for crisp closed types)
- **Twisted arrows** $\mathfrak{t}$: $\langle \mathfrak{t} \mid A \rangle$ has as $n$-simplices:

$$\begin{array}{ccccccccc} a_n & \longleftarrow & \dots & \longleftarrow & a_2 & \longleftarrow & a_1 & \longleftarrow & a_0 \\ \downarrow & & & & & & & & \\ a_{n+1} & \longrightarrow & \dots & \longrightarrow & a_{2n-2} & \longrightarrow & a_{2n-1} & \longrightarrow & a_{2n} \end{array}$$

**Mode theory:**

$$\flat \circ \flat = \flat \circ \mathfrak{o} = \flat \circ \sharp = \flat \qquad \sharp \circ \flat = \sharp \circ \mathfrak{o} = \sharp \circ \flat = \sharp$$

$$\mathfrak{o} \circ \mathfrak{o} = \mathrm{id} \qquad \flat \leq \mathrm{id} \leq \sharp \qquad \flat \leq \mathfrak{t}$$

# Axioms for triangulated type theory I

**Axiom (Interval $\mathbb{I}$)**

*There is a bounded distributive lattice* $(\mathbb{I} : \mathrm{Set}, 0, 1, \vee, \wedge)$

**Axiom (Path type former as modality)**

*The path type* $(-)^{\mathbb{I}}$ *is presented by a modality* $\mathfrak{p}$.

**Axiom (Crisp induction)**

*Modalities commute with path types: for every* $\mu$*, the map* $\mathrm{mod}_{\mu}(a) = \mathrm{mod}_{\mu}(b) \to \langle \mu \mid a = b \rangle$ *is an equivalence.*

**Axiom (Reversal on $\mathbb{I}$)**

*There is an equivalence* $\neg : \langle \mathfrak{o} \mid \mathbb{I} \rangle \to \mathbb{I}$ *which swaps* $0$ *for* $1$ *and* $\wedge$ *for* $\vee$.

# Axioms for triangulated type theory II

**Axiom ($\mathbb{I}$ detects discreteness)**

*If* $A :_\flat \mathcal{U}$ *then* $\langle \flat \mid A \rangle \to A$ *is an equivalence if and only if* $A \to (\mathbb{I} \to A)$ *is an equivalence.*

**Axiom (Global points of $\mathbb{I}$)**

*The map* $\mathrm{Bool} \to \mathbb{I}$ *is injective and induces an equivalence* $\mathrm{Bool} \simeq \langle \flat \mid I \rangle$.

**Axiom (Cubes separate)**

$f :_\flat A \to B$ *is an equivalence if and only if the following holds:*

$$\Pi_{n :_\flat \mathbb{N}} \, \mathrm{isEquiv} \left( (f_*)^\dagger : \langle \flat \mid \mathbb{I}^n \to A \rangle \to \langle \flat \mid \mathbb{I}^n \to B \rangle \right)$$

**Axiom (Simplicial stability)**

*If* $A :_\flat \mathcal{U}$ *then for all* $n :_\flat \mathbb{N}$ *the following map is an equivalence:*

$$\eta_* : \langle \flat \mid \Delta^n \to A \rangle \to \langle \flat \mid \Delta^n \to \boxslash A \rangle$$

# Axioms for triangulated type theory III

**Axiom (Twisted arrows)**

*For every category* $C :_\flat \mathcal{U}$ *we have morphisms* $\pi_0^{\mathrm{tw}} : \langle \mathfrak{t} \mid C \rangle \to \langle \mathfrak{o} \mid A \rangle$, $\pi_1^{\mathrm{tw}} : \langle \mathfrak{t} \mid C \rangle \to A$*, equivalences* $\iota : \langle \flat \mid (\langle \mathfrak{o} \mid \Delta^n \rangle \diamond \Delta^n) \to C \rangle \simeq \langle \flat \mid \Delta^n \to \langle \mathfrak{t} \mid C \rangle \rangle$ *and* $\tau : \langle \mathfrak{t} \mid C \rangle \simeq \langle \mathfrak{t} \mid \langle \mathfrak{o} \mid C \rangle \rangle$*, satisfying appropriate naturality and compatibility conditions.*

Here, $X \diamond Y$ is the *blunt join* $X \amalg_{X \times \{0\} Y} (X \times \mathbb{I} \times Y) \amalg_{X \times \{1\} Y} Y$.

**Axiom (Blechschmidt duality)**

*Let* $A$ *be a finitely presented* $\mathbb{I}$*-algebra, i.e.,* $A \simeq \mathbb{I}[x_1, \ldots, x_n]/(r_1 = s_1, \ldots, r_n = s_n)$*, then the evaluation map is an equivalence:*

$$\lambda a, f.f(a) : A \simeq (\hom_{\mathbb{I}}(A, \mathbb{I}) \to \mathbb{I})$$

# Simplicial vs cubical models

**Theorem (Kapulkin–Voevodsky '18, Sattler '18, Streicher–W '19)**

*Simplicial sets are an essential subtopos of cubical sets.*

Crucial for internal treatment of universes (jww Gratzer–Buchholtz).

Applications to model structures for $\infty$-categories (Hackney–Rovelli, Cavallo–Sattler)

# Towards the universe of spaces

- Covariant families have **transport**: $(-)_! : \prod_{a,b:X} (a \to_X b) \to A(a) \to A(b)$
- If $X$ is Segal, then each fiber $A(a)$ is discrete.
- Can we take $\sum_{A:\mathcal{U}} \mathrm{isCov}(A)$?
- **No**: $\mathrm{isCov}(A)$ just means that $A$ is discrete; doesn't see variance.
- Need a predicate that yields covariance over all possible contexts.
- **Solution: Amazing fibrations** due to M. Riley (2024): *A Type Theory with a Tiny Object* , arXiv:2403.01939; based on Licata–Orton–Pitts–Spitters '18 (which was used for similar purposes by Weaver–Licata '20)

# Amazingly covariant families

- Consider $\mathrm{isCov}(A : \mathbb{I} \to \mathcal{U}) \simeq \prod_{x:A(0)} \mathrm{isContr}\left(\sum_{y:A(1)} (x \to_\alpha y)\right)$, where $\alpha : \mathrm{hom}_{\mathbb{I}}(0,1)$.
- This gives a predicate $\mathrm{isCov}_{\mathbb{I}} : \mathcal{U}^{\mathbb{I}} \to \mathrm{Prop}$.

**Definition (Amazingly covariant types)**

Let $A : \mathcal{U}$ be a type. It is *amazingly covariant* if and only if the following proposition is inhabited:

$$\mathrm{isACov}(A) :\equiv \big(\mathrm{isCov}_{\mathbb{I}}(\lambda i. A^{\eta}(i))\big)_{\mathbb{I}},$$

where $A^{\eta}$ is the image of $A$ under the unit $\eta_{\mathcal{U}} : \mathcal{U} \to (\mathcal{U}^{\mathbb{I}})_{\mathbb{I}}$.

# The universe of spaces

The simplicial objects give rise to the (simplicial) subuniverse of simplicial types:

$$\mathcal{U}_{\boxslash} :\equiv \sum_{A:\mathcal{U}} \mathrm{isSimp}(A)$$

**Definition**

We call $\mathcal{S} :\equiv \sum_{A:\mathcal{U}_{\boxslash}} \mathrm{isACov}(A)$ the **universe of spaces**.

**Theorem**

1. *The universe $\mathcal{S}$ is a synthetic $\infty$-category whose terms are $\infty$-groupoids.*
2. *$\mathcal{S}$ classifies amazingly covariant families in $\mathcal{U}_{\boxslash}$.*
3. *$\mathcal{S}$ is closed under $\Sigma$, identity types, and finite (co)limits.*
4. *$\mathcal{S}$ is **directed univalent**:*

$$\mathrm{arrtofun} : (\Delta^1 \to \mathcal{S}) \simeq \left( \sum_{A,B:\mathcal{S}} (A \to B) \right)$$

# Equivalence lemma

**Theorem**

*Assume maps $f, g : \Delta^1 \to \mathcal{S}$ and a natural transformation $\alpha : \prod_{x:\Delta^1} f(x) \to g(x)$. Then $\alpha$ is a family of equivalences if and only if $\alpha(0)$ and $\alpha(1)$ are equivalences.*

$$\begin{array}{ccc} f\,0 & \xrightarrow[\cong]{\alpha\,0} & g\,0 \\ \downarrow & & \downarrow \\ f\,1 & \xrightarrow[\alpha\,1]{\cong} & g\,1 \end{array}$$

For the proof, we need the axiom that cubes detect equivalences:

$$\left( \prod_{n:\mathrm{Nat}} \langle \flat \mid \mathbb{I}^n \to A \rangle \simeq \langle \flat \mid \mathbb{I}^n \to B \rangle \right) \to (A \simeq B)$$

We can also prove a generalization of the equivalence lemma to $\Delta^n$.

# Directed univalence

1. Since $\mathcal{S}$ classifies (amazingly) covariant families, there is a map

$$\mathrm{arrtofun} :\equiv \lambda F.(F\,0, F\,1, \alpha_!^F : F\,0 \to F\,1) : (\Delta^1 \to \mathcal{S}) \to \Big( \sum_{A,B:\mathcal{S}} (A \to B) \Big).$$

2. In the other direction, we consider the **mapping cone/directed glue type** (cf. cubical type theory and Weaver–Licata '20):

$$\mathrm{Gl} :\equiv A, B, f.\lambda i. \sum_{b:B} (i = 0) \to f^{-1}(b) : \Big( \sum_{A,B:\mathcal{S}} (A \to B) \Big) \to (\Delta^1 \to \mathcal{S})$$

3. We show that they form an inverse pair making crucial use of the equivalence lemma.
4. Segalness of $\mathcal{S}$ is using similar arguments, but in higher dimensions.

Analogous result in bicubical setting by Weaver–Licata '20, but some difference in methods and axioms.

# Application: directed structure identity principle (DSIP)

**Theorem (DSIP for pointed spaces)**

*Let* $\mathcal{S}_* :\equiv \sum_{A:\mathcal{S}} A$*. Then for* $(A,a),(B,b) : \mathcal{S}_*$ *we have:*

$$\hom_{\mathcal{S}_*}((A,a),(B,b)) \simeq \sum_{f:A\to B} f(a) = b$$

**Theorem (DSIP for monoids)**

*Consider the type (category!) of (set-)monoids*

$$\mathrm{Monoid} :\equiv \sum_{A:\mathcal{S}_{\leq 0}} \sum_{\varepsilon:A} \sum_{\cdot:A\times A} \mathrm{isAssoc}(\cdot) \times \mathrm{isUnit}(\cdot,\varepsilon).$$

*Then homomorphisms from* $(A,\varepsilon_A,\cdot_A,\alpha_A,\mu_A)$ *to* $(B,\varepsilon_B,\cdot_B,\alpha_B,\mu_B)$ *correspond to set maps* $A\to B$ *compatible with multiplication and units.*

# Towards synthetic higher algebra

We can internally define presheaf categories $\mathrm{PSh}(C) :\equiv \langle \mathfrak{o} | C \rangle \to \mathcal{S}$.

**Definition (∞-monoids)**

The category $\mathrm{Mon}_\infty$ of ∞*-monoids* is the full subcategory[a] of $\mathrm{PSh}(\Delta)$ defined by the predicate

$$\varphi(X :_\flat \mathrm{PSh}(\Delta)) :\equiv \prod_{n:\mathrm{Nat}} \mathrm{isEquiv}(\langle X(\iota_k)_{k<n} \rangle : X(\Delta^n) \to X(\Delta^1)^n)$$

[a] need the codiscrete modality ♯

This encodes the structure of a homotopy-coherent monoid. Multiplication is given through

$$\mu_X : X(\Delta^2) \simeq X(\Delta^1)^2 \to X(\Delta^1).$$

**Definition (∞-groups)**

The category $\mathrm{Grp}_\infty$ of ∞*-groups* is the full subcategory of $\mathrm{Mon}_\infty$ defined by the predicate

$$\varphi(X :_\flat \mathrm{Mon}_\infty) :\equiv \mathrm{isEquiv}(\lambda x, y. \langle x, \mu_X(x,y) \rangle : X(\Delta^1)^2 \to X(\Delta^1)^2)$$

One can show that both these categories satisfy the expcted DSIP.

# The category of spectra

## Definition (The category of spectra)

The type of *spectra* is defined as the limit (in the ambient universe)

$$\mathrm{Sp} :\equiv \varprojlim(\mathcal{S}_* \xleftarrow{\Omega} \mathcal{S}_* \xleftarrow{\Omega} \dots).$$

## Proposition

$\mathrm{Sp}$ *is a stable* $\infty$*-category and cocomplete.*

# Categorical Yoneda lemma

Let $C$ be a category. Using the twisted arrow modality $\mathfrak{t}$, we obtain the hom-bifunctor $\Phi : C \times \langle \mathfrak{o} | C \rangle \to \mathcal{S}$. We write $\mathbf{y}(c) :\equiv \Phi(-, c)$.
We now recover the synthetic $\infty$-categorical version of the "standard" Yoneda lemma:

**Theorem (Yoneda lemma)**

*We have* $\hom(\mathbf{y}(c), X) \simeq X(c)$*, naturally in each* $c :_\flat C$ *and* $X :_\flat \mathrm{PSh}(C)$*.*

**Theorem (Density)**

*If* $X :_\flat \mathrm{PSh}(C)$*, then* $X \simeq \varinjlim_{\langle \mathfrak{o} \mid \widetilde{X} \rangle} \mathbf{y} \circ \pi^{\mathrm{op}}$*.*

# Universal property of presheaf categories

**Theorem (Descent for presheaf categories)**

*Let* $E :\equiv \mathrm{PSh}(A)$ *and* $F :_\flat C \to E$*, then* $E/\varinjlim_{c:C} F(c) \simeq \varprojlim_{c:C} E/F(c)$.

**Theorem (Universal property of $\mathrm{PSh}(C)$)**

$\mathrm{PSh}(C)$ *is the free cocompletion of* $C$*:* $\mathbf{y}^* : (\mathrm{PSh}(C) \to_{\mathrm{cc}} E) \to (C \to E)$

# Kan extensions

*The notion of Kan extensions subsumes all the other fundamental concepts of category theory.*

– S. Mac Lane '71

**Definition (Kan extensions)**

Given $f :_\flat C \to D$ and a category $E$, the left Kan extension $\mathrm{lan}_f$ is the left adjoint to $f^* : E^D \to E^C$.

**Theorem (Colimit formula)**

*If $E$ is cocomplete, then $\mathrm{lan}_f$ exists. For $X :_\flat C \to E$ it computes to*
$\mathrm{lan}_f X\, d \simeq \varinjlim(C \times_D D/d \to C \to E)$

# Cofinal functors

**Definition (Cofinal functors)**

A functor $f :_\flat C \to D$ is *right cofinal* if for every $X :_\flat D \to \mathcal{S}$ the map $\varprojlim_D X \to \varprojlim_C X \circ f$ is an equivalence.

**Proposition (Characterization of right cofinality)**

*Let $f :_\flat C \to D$ be a functor. Then the following are equivalent:*

1. *$f$ is right cofinal.*
2. *Let $X :_\flat \langle \mathfrak{o} \mid A \rangle \to \mathcal{S}$ be family with associated right fibration $\pi :_\flat \widetilde{X} \to A$. Then any square of the following form has a filler $\overline{\varphi}$, uniquely up to homotopy:*

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & \widetilde{X} \\ {\scriptstyle f}\downarrow & \overset{\overline{\varphi}}{\nearrow} & \downarrow{\scriptstyle \pi} \\ D & \xrightarrow[\alpha]{} & A \end{array}$$

3. *$f$ is a contravariant equivalence, i.e., for all $p :_\flat C \to A$ and $q :_\flat D \to A$ with $q \circ f = p$, we have that: $f^* :_\flat (\Pi_{a:A} D_a \to X_a) \xrightarrow{\simeq} (\Pi_{a:A} C_a \to X_a)$*

# Quillen's Theorem A

**Theorem (Quillen's Theorem A)**

*A functor* $f :_\flat C \to D$ *is right cofinal if and only if* $L_{\mathbb{I}}(C \times_D d/D) \simeq \mathbf{1}$ *for each* $d :_\flat D$.

This follows by reducing to the case of presheaves and ultimately groupoids $\mathcal{S}$.

# Application to cocartesian fibrations

**Theorem (Properness of cocartesian fibrations)**

*As below, if $\pi$ are cocartesian and $u$ is right cofinal then $v$ is right cofinal:*

$$\begin{array}{ccc} A \times_B E & \xrightarrow{\quad v \quad} & E \\ {\scriptstyle \xi}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \pi} \\ A & \xrightarrow{\quad u \quad} & B \end{array}$$

Using Quillen's Theorem A and some localization theory we can give a new synthetic proof:

**Proof.**

We compute the fiber:

$$(A \times_B E) \times_E e/E \simeq A \times_B e/E \simeq A \times_B \left(\Sigma_{b':B} \Sigma_{f:(\pi(e) \to_B b')} (E_{b'}^{\Delta^1})\right)$$
$$\simeq \Sigma_{\langle a,f \rangle : A \times_B \pi(e)/B} \; f_! \, e/E_{u(a)}$$

Now, we have both $L_{\mathbb{I}}(A \times_B \pi(e)/B) \simeq \mathbf{1}$ and $L_{\mathbb{I}}(f_! \, e/E_{u(a)}) \simeq \mathbf{1}$. This suffices by a theorem in: E. Rijke, M. Shulman, B. Spitters (2020): *Modalities in homotopy type theory*. □

# Outlook

1. Synthetic higher algebra
2. Universe of higher categories
3. Extend formalizations
4. ...

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