

What is a successor within an ordinal, constructively?

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HoTTTEST

October 22, 2026

Ordinals and simulations

Recall definitions from §10.3 of the HoTT book:

- ▶ An **ordinal** is a **type** A together with a relation $<: A \rightarrow A \rightarrow \text{Prop}$ which is
 - ▶ *transitive*: $\forall(x, y, z : A). x < y \rightarrow y < z \rightarrow x < z$;
 - ▶ *extensional*: $(\forall(z : A). z < x \leftrightarrow z < y) \rightarrow x = y$ for any $x, y : A$; and
 - ▶ *well-founded*: every element of A is *accessible*.

Write Ord for the type of ordinals.

- ▶ A **simulation** from ordinal $(A, <_A)$ to $(B, <_B)$ is a map $f : A \rightarrow B$ such that
 - ▶ f is order-preserving: $a_1 <_A a_2$ implies $f a_1 <_B f a_2$ for all $a_1, a_2 : A$; and
 - ▶ for all $a : A$ and $b : B$, if $b <_B f a$, then we have an $a_0 <_A a$ with $f a_0 = b$.

Write $A \leq B$ for the **type** of simulations from $(A, <_A)$ to $(B, <_B)$.

The ordinal of ordinals

- ▶ Given an ordinal $(A, <_A)$ and $a : A$, the **initial segment** of elements below a is given as $A_{/a} \equiv \Sigma(x : A). x <_A a$.
 - ▶ With $<_A$ inherited from A , the type $A_{/a}$ forms an ordinal.

Recall results from **TypeTopology**:

- ▶ Given ordinals A and B , define $A < B \equiv \Sigma(b : B). A = B_{/b}$.
 - ▶ $A \leq B \leftrightarrow (\forall (C : \text{Ord}). C < A \rightarrow C < B)$
 - ▶ Univalence identifies isomorphic initial segments.
- ▶ $(\text{Ord}, <)$ is an ordinal, living in the next universe.
 - ▶ Univalence is needed in the extensionality proof.

The usual successor within Ord

The proof of Lemma 10.3.21 of the HoTT book defines, for $A : \mathbf{Ord}$,

$$S \equiv A + 1, \text{ with } \star : 1 \text{ being the greatest element,}$$

and calls S the **successor** of A .

Does it satisfy the familiar, classical properties of successor?

► $A < X \rightarrow S \leq X$ ✓

► $X < S \leftrightarrow (X < A) + (X = A)$ ✓

► $X < S \leftrightarrow X \leq A$ ✗ Take $A = 1$ and X to be a proposition P :

$$P \leq 1, \quad P < 1 + 1 \iff (P < 1) + (P = 1) \iff \neg P + P$$

Another construction of successors within Ord

Following Taylor¹, we call

$$S' \equiv \Sigma(Z : \text{Ord}). Z \leq A$$

the **fat successor** of A , which lives in the next universe of A .²

- ▶ $X < S' \leftrightarrow X \leq A$ ✓
- ▶ $X < S' \leftrightarrow (X < A) + (X = A)$ ✗

¹Paul Taylor. *Intuitionistic Sets and Ordinals*. Journal of Symbolic Logic 61(3), 1996.

²Using propositional truncation and propositional resizing, Tom de Jong constructs an equivalent ordinal in the same universe as the input in [TypeTopology](#).

Questions for today

1. What properties justify calling an element a successor?
2. What further properties might we expect?
3. How do these properties relate constructively?
4. Which properties are needed for which mathematical purposes?
5. Which ordinals admit successor operations with these properties, and what prevents stronger requirements from being satisfied constructively?

The general setting

Fix an ordinal $(A, <)$.

We study properties expressing that $s : A$ is a successor of $x : A$.

Consider two weak orders: for $x, y : A$, define

- ▶ Canonical weak order $\leq$

$$x \leq y \equiv \forall (z : A). z < x \rightarrow z < y$$

- ▶ Positive weak order $\leq^+$

$$x \leq^+ y \equiv (x < y) + (x = y)$$

Immediately-above: a minimal requirement

$$s \text{ isImmediatelyAbove } x \equiv (x < s) \times \neg(\Sigma(z : A). (x < z) \times (z < s))$$

- ▶ Minimal requirement: s lies above x , with no element between them.
- ▶ **Limitation**: for a fixed x , uniqueness of s does not follow constructively.

Least-above: an order-theoretic successor

$$s \text{ isLeastAbove } x \equiv (x < s) \times (\forall (z : A). x < z \rightarrow s \leq z)$$

- ▶ Implies $s \text{ isImmediatelyAbove } x$.
- ▶ $(X + 1) \text{ isLeastAbove } X$ ✓
 $(\Sigma C. C \leq X) \text{ isLeastAbove } X \rightarrow \text{EM}$
- ▶ s is unique whenever it exists.
- ▶ **Limitation**: leastness alone does not constructively rule out being a limit.

$s \text{ isLeastAbove } x$

 $s \text{ isImmediatelyAbove } x$

Can a least-above element also be a limit?

For $a : A$, define

$$\text{isLeastAboveElement}(a) \equiv \exists(b : A). a \text{ isLeastAbove } b$$

$$\text{isLimit}(a) \equiv \|A_{/a}\| \times (\forall b. (\forall z. z < a \rightarrow z \leq b) \rightarrow a \leq b)$$

Theorem. Suppose that, for every ordinal $A : \text{Ord}$ and every $a : A$, the type

$$\text{isLeastAboveElement}(a) + \text{isLimit}(a)$$

is a proposition. Then **countable double-negation shift** holds:

$$\forall(Q : \mathbb{N} \rightarrow \text{Prop}). (\forall n. \neg\neg Q_n) \rightarrow \neg\neg(\forall n. Q_n).$$

Greatest-below: excluding limits

$$x \text{ isGreatestBelow } s \equiv (x < s) \times (\forall (z : A). z < s \rightarrow z \leq x)$$

- ▶ Implies $s \text{ isImmediatelyAbove } x$.
- ▶ Excludes the limit case: the following type is a proposition
$$(\Sigma (x : A). x \text{ isGreatestBelow } a) + \text{isLimit}(a)$$
- ▶ `TypeTopology` uses this notion to define order limit points.
- ▶ Both $A \text{ isGreatestBelow } (A + 1)$ and $A \text{ isGreatestBelow } (\Sigma C. C \leq A)$ hold.
- ▶ **Limitation**: for a fixed x , uniqueness of s does not follow constructively.

$$\begin{array}{ccc} x \text{ isGreatestBelow } s & & s \text{ isLeastAbove } x \\ & \Downarrow & \Downarrow \\ & s \text{ isImmediatelyAbove } x & \end{array}$$

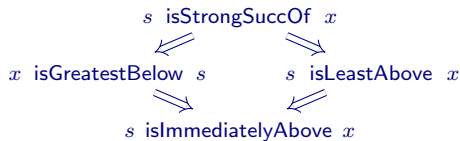
Strong successor: combining greatest-below and least-above

$$s \text{ isStrongSuccOf } x \equiv (x \text{ isGreatestBelow } s) \times (s \text{ isLeastAbove } x)$$

- Implies both $x \text{ isGreatestBelow } s$ and $s \text{ isLeastAbove } x$.
- Each of x and s uniquely determines the other.
- Supports a proposition-valued zero–successor–limit classification

$$\text{isZero}(a) + (\Sigma(x : A). a \text{ isStrongSuccOf } x) + \text{isLimit}(a)$$

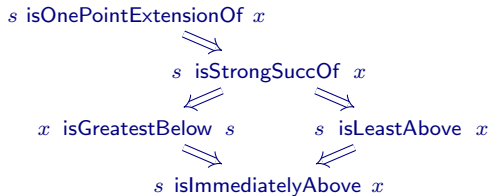
- $(X + 1) \text{ isStrongSuccOf } X$ ✓
- $(\Sigma C. C \leq X) \text{ isStrongSuccOf } X \rightarrow \text{EM}$



One-point extension: adding one element to an initial segment

$$s \text{ isOnePointExtensionOf } x \equiv \forall (z : A). z < s \leftrightarrow z \leq^+ x$$

- Implies $s \text{ isStrongSuccOf } x$.
- The corresponding initial segments satisfy $A_{/s} = A_{/x} + \mathbf{1}$.
- $(A + \mathbf{1}) \text{ isOnePointExtensionOf } A$ holds.
- If A has a 0 element, then successive one-point extensions give $1, 2, 3, \dots$ (finite elements).

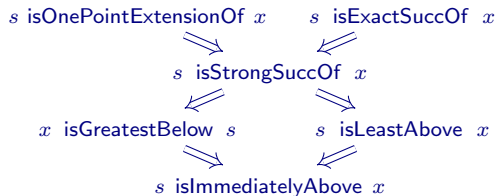


Exact successor: compatibility of the two orders

$$s \text{ isExactSuccOf } x \equiv \forall (z : A). (z < s \leftrightarrow z \leq x) \times (x < z \leftrightarrow s \leq z)$$

- Implies $s \text{ isStrongSuccOf } x$.
- $(X + 1) \text{ isExactSuccOf } X \rightarrow \text{EM}$
- If every element has an exact successor, then **mixed transitivity** holds:

$$\forall (x, y, z : A). x \leq y < z \rightarrow x < z$$



Exact successor function

$$s \text{ isExactSuccOf } x \equiv \forall(z : A). (z < s \leftrightarrow z \leq x) \times (x < z \leftrightarrow s \leq z)$$

$$\text{isExactSucc}(f) \equiv \forall(x : A). (fx) \text{ isExactSuccOf } x$$

Theorem. Every exact successor function $f : A \rightarrow A$ preserves both $<$ and $\leq$.

Proof. $<$ -preserving:

$$x < y \implies fx \leq y \implies fx < fy.$$

$\leq$ -preserving: Assume $x \leq y$. For every z ,

$$z < fx \implies z \leq x \implies z \leq y \implies z < fy;$$

thus $fx \leq fy$.

Order-preservation and mixed transitivity

$$s \text{ isLeastAbove } x \equiv (x < s) \times (\forall(z : A). x < z \rightarrow s \leq z)$$

$$\text{calcLeastAbove}(f) \equiv \forall(x : A). (fx) \text{ isLeastAbove } x$$

$$\text{MixedTransitivity} \equiv \forall(x, y, z : A). x \leq y < z \rightarrow x < z$$

Theorem. If A has a function $f : A \rightarrow A$ calculating least-above elements, then

$$f \text{ preserves } \leq \iff \text{MixedTransitivity}.$$

Proof. Mixed transitivity gives

$$x \leq y \implies x < fy \implies fx \leq fy.$$

Weak monotonicity gives

$$x \leq y < z \implies x < fx \leq fy \leq z \implies x < z.$$

When do the notions coincide?

For a fixed pair $x, s : A$.

Under **mixed transitivity**

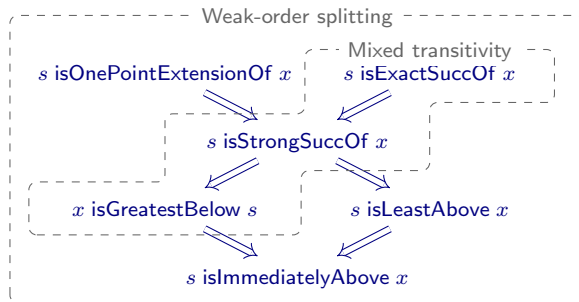
$$\forall u, v, w. u \leq v < w \rightarrow u < w,$$

greatest-below, strong and exact successors are equivalent.

Under **weak-order splitting**

$$\forall u, v. u \leq v \rightarrow u \leq^+ v,$$

all six notions are equivalent.



Cantor normal forms as a subset of binary trees

In classical set theory, ordinal α can be written uniquely in Cantor normal form

$$\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_n} \text{ with } \beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$$

Let **Cnf** be the ordinal of binary trees built using the constructors 0 and $\omega^a + b$, subject to Cantor normal form condition, with the hereditary lexicographic order.

Successors are computed recursively:

$$S(0) = \omega^0 + 0, \quad S(\omega^a + b) = \omega^a + S(b).$$

Hereditary lexicographic comparison is trichotomous.

The successor within **Cnf** satisfies all six notions.

Brouwer trees as a quotient inductive-inductive type

Define a set Brw and a proposition-valued order $\leq$ simultaneously.

Constructors for Brw

$$\mathsf{zero} : \mathsf{Brw}, \quad \mathsf{succ} : \mathsf{Brw} \rightarrow \mathsf{Brw}, \quad \mathsf{limit}(f) : \mathsf{Brw}$$

where $f : \mathbb{N} \rightarrow \mathsf{Brw}$ is strictly increasing.

A path constructor identifies the limits of mutually cofinal sequences.

This gives an ordinal $(\mathsf{Brw}, <)$, with $x < y \equiv \mathsf{succ}(x) \leq y$.

- ▶ $\mathsf{succ}(x)$ is $\mathsf{ExactSuccOf}$ x for every $x : \mathsf{Brw}$.
- ▶ succ gives one-point extensions at every finite element $\bar{n} \equiv \mathsf{succ}^n(0)$.

Brouwer trees: one-point extension and LPO

General result. Suppose $S : A \rightarrow A$ assigns exact successors. Then

$$(\forall(x : A). Sx \text{ isOnePointExtensionOf } x) \iff (\forall(u, v : A). u \leq v \rightarrow u \leq^+ v).$$

For Brouwer trees, weak-order splitting is equivalent to **LPO**.

LPO: every binary sequence is either constantly 0 or contains a 1.

Since **succ** is exact,

$$(\forall x : \text{Brw. succ}(x) \text{ isOnePointExtensionOf } x) \iff \text{LPO}.$$

One-point extension at every element is equivalent to **LPO**.

Taylor's plump ordinals

An ordinal A is **plump** [Taylor, 1996] if, recursively:

- ▶ Every strict predecessor of A is plump.
- ▶ If C is plump and $C \leq B < A$, then $C < A$.

These form an ordinal **Plump**, with the order inherited from **Ord**.

Using propositional truncation and resizing, take a small representative

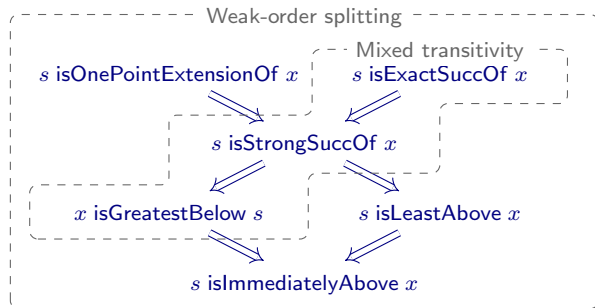
$$S(A) \simeq \Sigma(C : \mathbf{Ord}). \text{isPlump}(C) \times (C \leq A).$$

This is plump, and S assigns **exact successors** within **Plump**.

One-point extension at every element is equivalent to **excluded middle**:

$$(\forall A : \mathbf{Plump}. S(A) \text{isOnePointExtensionOf } A) \iff \text{EM}.$$

Summary and question



Ordinal	One-point	Exact
Cnf	✓	✓
Brw	LPO	✓
Plump	EM	✓
Ord	✓	EM

Which successor properties would you want in your own mathematical work?